ASSIGNMENT 4 Math 245 (Winter 2009)

Due at the start of class on Wednesday 4 February.

1. Let $\mathsf{T}_1 \in \mathcal{L}(\mathsf{V}_1)$ and $\mathsf{T}_2 \in (\mathsf{V}_2)$, where $\mathsf{V}_1, \mathsf{V}_2$ are vector spaces over a field $\mathbb{F}$. The *spectrum* of a linear transformation T is $\operatorname{spec}(\mathsf{T}) := \{\lambda \in \mathbb{F} : \lambda \text{ is an eigenvalue of } \mathsf{T}\}$.
   (a) Prove that if $\lambda \in \operatorname{spec}(\mathsf{T}_1)$ or $\lambda \in \operatorname{spec}(\mathsf{T}_2)$, then $\lambda \in \operatorname{spec}(\mathsf{T}_1 \oplus \mathsf{T}_2)$.
   (b) Prove or disprove: $\operatorname{spec}(\mathsf{T}_1 \oplus \mathsf{T}_2) = \operatorname{spec}(\mathsf{T}_1) \cup \operatorname{spec}(\mathsf{T}_2)$.
   (c) Prove that if $\lambda_1 \in \operatorname{spec}(\mathsf{T}_1)$ and $\lambda_2 \in \operatorname{spec}(\mathsf{T}_2)$, then $\lambda_1\lambda_2 \in \operatorname{spec}(\mathsf{T}_1 \otimes \mathsf{T}_2)$.
   (d) Prove or disprove: $\operatorname{spec}(\mathsf{T}_1 \otimes \mathsf{T}_2) = \{\lambda_1\lambda_2 : \lambda_1 \in \operatorname{spec}(\mathsf{T}_1) \text{ and } \lambda_2 \in \operatorname{spec}(\mathsf{T}_2)\}$.

2. Compute the eigenvalues and corresponding eigenspaces of the following matrices:

$$A = \begin{pmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -2 & -4 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 0 \\ 8 & 25 & -16 \\ 14 & 43 & -27 \end{pmatrix}.$$

3. In this problem, all matrices are from $\mathsf{M}_{n\times n}(\mathbb{C})$ with $n \geq 2$.
   (a) Compute the characteristic polynomial of

$$C_n := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 \\ 0 & 0 & 0 & 1 & \ddots & \vdots \\ \vdots & 0 & 0 & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

   (b) What are the eigenvalues of C_n?
   (c) Compute the eigenvalues of

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_0 & a_1 & a_2 & & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & a_1 & \ddots & \vdots \\ \vdots & a_{n-2} & a_{n-1} & \ddots & \ddots & a_2 \\ a_2 & & \ddots & \ddots & a_0 & a_1 \\ a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} & a_0 \end{pmatrix}$$

   for arbitrary $a_0, \ldots, a_{n-1} \in \mathbb{C}$. (Such a matrix is called a *circulant matrix*.)

4. Let V be a finite-dimensional vector space over a field $\mathbb{F}$. We say that $\mathsf{T}, \mathsf{U} \in \mathcal{L}(\mathsf{V})$ are *simultaneously diagonalizable* when there is a single basis β of V such that $[\mathsf{T}]_\beta, [\mathsf{U}]_\beta$ are both diagonal. We say that T, U *commute* when $\mathsf{TU} = \mathsf{UT}$.
   (a) Prove that if T, U are simultaneously diagonalizable, then they commute.

(b) Prove that if T, U commute and each is diagonalizable, then T, U are simultaneously diagonalizable.

(c) Show that the matrices

$$A = \begin{pmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -2 & -4 & 3 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -2 & 2 \end{pmatrix}$$

commute, and find an invertible matrix S such that $S^{-1}AS, S^{-1}DS$ are both diagonal.

5. An *adjacency matrix* is a square, symmetric zero-one matrix with all diagonal entries equal to 0. A *permutation matrix* is a square zero-one matrix with exactly one 1 in each row and in each column. Two adjacency matrices A_1, A_2 are said to be *isomorphic* when there exists some permutation matrix P such that $A_2 = P^t A_1 P$.

   (a) Prove that if A_1 and A_2 are isomorphic, then $\operatorname{spec}(A_1) = \operatorname{spec}(A_2)$ (i.e., they have the same eigenvalues).

   (b) Prove or disprove the converse of (a).