ASSIGNMENT 3 Math 245 (Winter 2009)

Due at the start of class on Wednesday 28 January.

1. The *permanent* of $A \in \mathsf{M}_{n\times n}(\mathbb{F})$ is defined as

$$\mathrm{per}(A) := \sum_{j=1}^{n} A_{1j}\,\mathrm{per}(\tilde{A}_{1j}),$$

where $\tilde{A}_{ij}$ denotes the matrix obtained by deleting row i and column j from A, and where the permanent of a scalar is that scalar.

   (a) Prove that the permanent is row-multilinear.

   (b) Show that

$$\mathrm{per}(A) = \sum_{j=1}^{n} A_{ij}\,\mathrm{per}(\tilde{A}_{ij})$$

   for any $i \in \{1, 2, \ldots, n\}$.

   (c) Show that if B is obtained by interchanging two rows of A, then $\mathrm{per}(B) = \mathrm{per}(A)$.

   (d) Suppose B is obtained by adding a scalar multiple of one row of A to another row of A. Can we conclude that $\mathrm{per}(B) = \mathrm{per}(A)$?

2. Let $A, B, C, D \in \mathsf{M}_{n\times n}(\mathbb{F})$.

   (a) Show that if any one of A, B, C, D is the zero matrix, then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC). \tag{$\star$}$$

   (b) Show that if C and D commute (i.e., if $CD = DC$), and if C is invertible, then $(\star)$ holds.

   (c) Does $(\star)$ hold for general A, B, C, D?

3. A matrix $\Pi \in \mathsf{M}_{n\times n}(\mathbb{R})$ is called a *projection matrix* if $\Pi^2 = \Pi$. A matrix $R \in \mathsf{M}_{n\times n}(\mathbb{R})$ is called a *reflection matrix* if $R = I - 2\Pi$ for some projection matrix Π, where I denotes the $n\times n$ identity matrix.

   (a) Show that if R is a reflection matrix, then $\det(R) = \pm 1$.

   (b) Let $v \in \mathsf{M}_{n\times 1}(\mathbb{R})$ satisfy $v \cdot v = 1$. Show that vv^t is a projection matrix (and hence that $I - 2vv^t$ is a reflection matrix).

   (c) Let v be as in part (b). Show that $\det(I - 2vv^t) = -1$.

   (d) Suppose $v, w \in \mathsf{M}_{n\times 1}(R)$ satisfy $v\cdot v = w\cdot w = 1$ and $v\cdot w = 0$. What is $\det(I - 2vv^t - 2ww^t)$?

4. The *trace* of a matrix $A \in \mathsf{M}_{n\times n}(\mathbb{C})$, denoted $\mathrm{tr}(A)$, is defined as the sum of its diagonal entries. The *exponential* of A is defined as the power series

$$e^A := \sum_{j=0}^{\infty} \frac{A^j}{j!}.$$

   (a) Show that if $U \in \mathsf{M}_{n\times n}(\mathbb{C})$ is an upper triangular matrix, then $\det(e^U) = e^{\mathrm{tr}(U)}$.

(b) Let $A, B \in \mathsf{M}_{n\times n}(\mathbb{C})$. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, and deduce that $\mathrm{tr}(A) = \mathrm{tr}(PAP^{-1})$ for any invertible matrix $P \in \mathsf{M}_{n\times n}(\mathbb{C})$.

(c) Show that if A is similar to an upper triangular matrix (i.e., if there exists an invertible matrix P such that $A = PUP^{-1}$ for some upper triangular matrix U), then $\det(e^A) = e^{\mathrm{tr}(A)}$.

(In fact, the identity $\det(e^A) = e^{\mathrm{tr}(A)}$ holds for *any* matrix $A \in \mathsf{M}_{n\times n}(\mathbb{C})$, since it can be shown that any complex matrix is similar to some upper triangular matrix.)