ASSIGNMENT 2 — Math 245 (Winter 2009)

Due at the start of class on Wednesday 21 January.

1. Let $A \in \mathsf{M}_{a\times a}(\mathbb{F})$ and $B \in \mathsf{M}_{b\times b}(\mathbb{F})$ for some field $\mathbb{F}$.
   (a) Prove that $\det(A \oplus B) = \det(A)\det(B)$.
   (b) Prove that $\det(A \otimes B) = \det(A)^b \det(B)^a$.

2. Compute $\det(C_n)$, where

$$C_n := \begin{pmatrix} 1 & i & 0 & \cdots & 0 \\ i & 1 & i & \ddots & \vdots \\ 0 & i & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & i \\ 0 & \cdots & 0 & i & 1 \end{pmatrix} \in \mathsf{M}_{n\times n}(\mathbb{C}).$$

3. Compute

$$\det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ b_1 & a_1 & a_1 & \cdots & a_1 & a_1 \\ b_1 & b_2 & a_2 & \cdots & a_2 & a_2 \\ b_1 & b_2 & b_3 & a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ b_1 & b_2 & b_3 & \cdots & b_n & a_n \end{pmatrix}.$$

4. We call $A \in \mathsf{M}_{n\times n}(\mathbb{R})$ a *zero-one matrix* when every entry is either 0 or 1. What is the largest number of 1s in a zero-one matrix of size $n \times n$ satisfying $\det(A) \neq 0$?

5. Let $f_1, f_2, \ldots, f_n \in \mathbb{F}[x]$, the vector space of polynomials in x with coefficients from the field $\mathbb{F}$. The determinant

$$\mathrm{Wr}(f_1, f_2, \ldots, f_n)(x) := \det \begin{pmatrix} f_1(x) & f_1'(x) & \cdots & f_1^{(n-1)}(x) \\ f_2(x) & f_2'(x) & \cdots & f_2^{(n-1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_n(x) & f_n'(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix},$$

   where a prime denotes differentiation with respect to x and a superscript (j) denotes the jth derivative, is called the *Wronskian* of $f_1, f_2, \ldots, f_n$. Prove that the polynomials $f_1, f_2, \ldots, f_n$ are linearly independent if and only if $\mathrm{Wr}(f_1, \ldots, f_n)(x)$ is nonzero (i.e., not the zero polynomial).