ASSIGNMENT 10 Math 245 (Winter 2009)

Due at the start of class on Wednesday 1 April.

1. In class, we analyzed a quantum Markov chain that searches for a single needle in a quantum haystack containing one needle and $n-1$ pieces of hay. In this problem, you will consider the case where the quantum haystack contains k needles and $n-k$ pieces of hay.

   (a) Construct a quantum Markov chain that describes the search, defined by a unitary matrix $U \in \mathsf{M}_{n\times n}(\mathbb{C})$. Your initial state should be $u \in \mathbb{C}^n$, where $u_i = 1/\sqrt{n}$ for all $i \in \{1, \ldots, n\}$.

   (b) Let $v \in \mathbb{C}^n$ be the quantum state with $v_i = 1/\sqrt{k}$ if state i represents a needle and $v_i = 0$ otherwise. Show that $W := \operatorname{span}\{u, v\}$ is a U-invariant subspace of $\mathbb{C}^n$, and construct an orthonormal basis for it.

   (c) Compute $U^m u$, where m is a positive integer. How should one choose m so that $|\langle U^m u, v\rangle|^2$ is close to 1?

2. Let $U \in \mathsf{M}_{n\times n}(\mathbb{C})$ be a unitary matrix. Recall that if $U \neq I$, then $\lim_{m\to\infty} U^m$ does not exist, so U does not have a unique limiting distribution. Nevertheless, in this problem we explore one way of defining a sensible notion of the limiting distribution of a quantum Markov chain.

   (a) Define the *limiting distribution* of the quantum Markov chain with initial state $v \in \mathbb{C}^n$ and unitary transition matrix $U \in \mathsf{M}_{n\times n}(\mathbb{C})$ to be the vector $p \in \mathbb{R}^n$ defined by

   $$p_j := \lim_{M\to\infty} \frac{1}{M} \sum_{m=0}^{M-1} |(U^m v)_j|^2.$$

   (You can think of this as running the quantum Markov chain for m steps, where m is chosen randomly between 0 and $M-1$, and then making a measurement.) Show that this limit always exists, and that p is a probability vector. In particular, give an expression for p in terms of v and the spectral decomposition of U.

   (b) Recall that if A is a regular stochastic matrix, then $\lim_{m\to\infty} A^m$ exists and has all columns identical, so that the corresponding Markov chain has a unique limiting distribution $A^m w$ that is independent of the initial probability vector w. Is the limiting distribution of a quantum Markov chain, as defined in part (a), independent of the initial quantum state v?

3. Let V be a finite-dimensional inner product space, and let $\mathsf{P}, \mathsf{Q} \in \mathcal{L}(\mathsf{V})$ be orthogonal projections.

   (a) Let $v \in \mathsf{V}$ be an eigenvector of $\mathsf{P}+\mathsf{Q}$. Show that the P-cyclic subspace generated by v is Q-invariant.

   (b) Prove *Jordan's Lemma*, which states that $\mathsf{V} = \bigoplus_i \mathsf{W}_i$ for some subspaces W_i of V, where $\dim(\mathsf{W}_i) \leq 2$, and each W_i is both P-invariant and Q-invariant.

   (c) Conclude that each W_i is $(\mathsf{I}-2\mathsf{Q})(\mathsf{I}-2\mathsf{P})$-invariant (i.e., is an invariant subspace of the product of reflections about the subspaces P and Q project onto).

4. Recall that the *minimal polynomial* of a matrix $A \in \mathsf{M}_{n\times n}(\mathbb{C})$ is the unique monic polynomial $p \in \mathbb{C}[x]$ of lowest degree such that $p(A)$ is the zero matrix.

(a) Compute the minimal polynomial of the matrix

$$A_{\lambda,n} := \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix} \in \mathsf{M}_{n\times n}(\mathbb{C}).$$

(b) What is the minimial polynomial of $A_{\lambda,n} \oplus A_{\mu,m}$?