ASSIGNMENT 1 Math 245 (Winter 2009)

Due at the start of class on Wednesday 14 January.

1. Let $\mathsf{V}_1, \mathsf{V}_2$ be vector spaces. Prove or disprove each of the following.

   (a) If W_1 is a subspace of V_1 and W_2 is a subspace of V_2, then $\mathsf{W}_1 \oplus \mathsf{W}_2$ is a subspace of $\mathsf{V}_1 \oplus \mathsf{V}_2$.

   (b) If W_1 is a subspace of V_1 and W_2 is a subspace of V_2, then $\mathsf{W}_1 \otimes \mathsf{W}_2$ is a subspace of $\mathsf{V}_1 \otimes \mathsf{V}_2$.

   (c) If W is a subspace of $\mathsf{V}_1 \oplus \mathsf{V}_2$, then there exists a subspace W_1 of V_1 and a subspace W_2 of V_2 such that $\mathsf{W} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

   (d) If W is a subspace of $\mathsf{V}_1 \otimes \mathsf{V}_2$, then there exists a subspace W_1 of V_1 and a subspace W_2 of V_2 such that $\mathsf{W} = \mathsf{W}_1 \otimes \mathsf{W}_2$.

   (e) If W_1 is a subspace of V_1 and W_2 is a subspace of V_2, then $(\mathsf{V}_1 \oplus \mathsf{V}_2)/(\mathsf{W}_1 \oplus \mathsf{W}_2) = \mathsf{V}_1/\mathsf{W}_1 \oplus \mathsf{V}_2/\mathsf{W}_2$.

   (f) If W_1 is a subspace of V_1 and W_2 is a subspace of V_2, then $(\mathsf{V}_1 \otimes \mathsf{V}_2)/(\mathsf{W}_1 \otimes \mathsf{W}_2) = \mathsf{V}_1/\mathsf{W}_1 \otimes \mathsf{V}_2/\mathsf{W}_2$.

2. Let V be an inner product space with $\dim \mathsf{V} \geq 2$. Call $\mathsf{T} \in \mathcal{L}(\mathsf{V} \otimes \mathsf{V})$ a *unit vector cloner* if there exists a fixed $w \in \mathsf{V}$ such that for all $v \in \mathsf{V}$ with $\langle v, v \rangle = 1$, $\mathsf{T}(v \otimes w) = v \otimes v$. Show that a unit vector cloner does not exist.

3. Define $A := \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $B := \begin{pmatrix} 3 & 3 & 2 \\ 5 & 4 & 3 \\ 0 & 3 & 1 \end{pmatrix}$.

   (a) Compute $\operatorname{rank}(A)$ and $\operatorname{rank}(B)$.

   (b) Compute $A \otimes B$.

   (c) Compute $\operatorname{rank}(A \otimes B)$.

   (d) Prove that for any two matrices C, D, $\operatorname{rank}(C \otimes D) = \operatorname{rank}(C) \operatorname{rank}(D)$.

4. Let A, B be matrices.

   (a) Prove that $A \oplus B$ is invertible if and only if A, B are invertible.

   (b) Is the same true of $A \otimes B$? Why or why not?

5. In this problem, we explore another way to make new vector spaces from old ones. Let V be a vector space with subspaces U, W.

   (a) Under what conditions is $\mathsf{U} \cap \mathsf{W}$ a subspace of V?

   (b) Supposing that V is finite-dimensional, give an expression (with proof) for $\dim(\mathsf{U} \cap \mathsf{W})$ in terms of $\dim \mathsf{U}$, $\dim \mathsf{W}$, and $\dim(\mathsf{U} + \mathsf{W})$.