THE MINIMALLY NON-IDEAL BINARY CLUTTERS WITH A TRIANGLE

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ABSTRACT. It is proved that the lines of the Fano plane and the odd circuits of K_5 constitute the only minimally non-ideal binary clutters that have a triangle.

1. INTRODUCTION

A *clutter* $\mathbb{F}$ over a finite *ground set* $E(\mathbb{F})$ is a family of subsets of $E(\mathbb{F})$ where no subset is contained in another one. We say that $R \subseteq E(\mathbb{F})$ is a *cover* of $\mathbb{F}$ if, for all $S \in \mathbb{F}$, $S \cap R \neq \emptyset$. The *blocker* $b(\mathbb{F})$ of $\mathbb{F}$ is the clutter, over the same ground set, of all (inclusion-wise) minimal covers of $\mathbb{F}$. It is well known that for any clutter $\mathbb{F}$, $b(b(\mathbb{F})) = \mathbb{F}$ [4, 8]. We say that $\mathbb{F}$ is *binary* if, for all $S \in \mathbb{F}$ and $R \in b(\mathbb{F})$, $|S \cap R|$ is odd. By definition, if a clutter is binary, then so is its blocker. Take disjoint subsets $I, J \subseteq E(\mathbb{F})$. Then $\mathbb{F}/I \setminus J$ denotes the clutter over ground set $E(\mathbb{F}) - (I \cup J)$ that consists of the minimal sets in $\{S - I : S \in \mathbb{F}, S \cap J = \emptyset\}$.[1] We say that $\mathbb{F}/I \setminus J$ is a *minor* of $\mathbb{F}$; it is a *proper minor* if $I \cup J \neq \emptyset$. It can be readily checked that if $\mathbb{F}$ is binary, then so are all its minors [17]. We say clutters $\mathbb{F}_1$ and $\mathbb{F}_2$ are *isomorphic*, and denote it by $\mathbb{F}_1 \cong \mathbb{F}_2$, if relabeling the ground set of $\mathbb{F}_1$ yields $\mathbb{F}_2$.

Denote by $\mathbb{L}_7$ the clutter of the lines of the Fano plane, that is,

$$\mathbb{L}_7 \cong \big\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{3,4,6\},\{3,5,7\},\{2,5,6\},\{2,4,7\}\big\}.$$

It can be readily checked that $\mathbb{L}_7 = b(\mathbb{L}_7)$ and that $\mathbb{L}_7$ is binary. A *cycle* in a graph is a non-empty edge subset where every vertex is incident to an even number of the edges, and a *circuit* is a minimal cycle. Denote by $\mathbb{O}_5$ the clutter, over ground set $E(K_5)$, of odd circuits of the complete graph K_5. The two clutters $\mathbb{O}_5, b(\mathbb{O}_5)$ are also binary.

A clutter $\mathbb{F}$ is *ideal* if the polyhedron $\left\{x \in \mathbb{R}_+^{E(\mathbb{F})} : x(S) \geq 1 \ \forall S \in \mathbb{F}\right\}$ has only integral extreme points. If a clutter is ideal, then so are all its minors [18]. A clutter is *minimally non-ideal (mni)* if it is not ideal and every proper minor is ideal. For instance, the three clutters $\mathbb{L}_7, \mathbb{O}_5$ and $b(\mathbb{O}_5)$ are mni. Notice that every non-ideal clutter has an mni clutter as a minor. Seymour ([18], page 200) proposed in 1977 the following conjecture:

The f-flowing conjecture. $\mathbb{L}_7, \mathbb{O}_5$ and $b(\mathbb{O}_5)$ are the only mni binary clutters.

A *triangle* in clutter $\mathbb{F}$ is a set $S \in \mathbb{F}$ such that $|S| = 3$. Observe that both $\mathbb{L}_7$ and $\mathbb{O}_5$ have triangles. As $b(b(\mathbb{O}_5)) = \mathbb{O}_5$, the f-flowing conjecture implies immediately the following:

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[1]Given sets A, B we denote by $A - B$ the set $\{a \in A : a \notin B\}$ and, for element a, we write $A - a$ instead of $A - \{a\}$.

The weak f-flowing conjecture. If $\mathbb{F}$ is an mni binary clutter, then $\mathbb{F}$ or $b(\mathbb{F})$ has a triangle.

The following is the main result of the paper:

Theorem 1. *$\mathbb{L}_7$ and $\mathbb{O}_5$ are the only mni binary clutters that have a triangle.*

In other words we prove that the *weak f-flowing conjecture* implies the *f-flowing conjecture*.

1.1. **Review of existing results.** All matroids considered in this paper are binary, and a basic knowledge of these matroids is assumed; we follow for the most part the notation used in Oxley [12] (second edition). Take a matroid M over ground set $E(M)$. Recall that a circuit is a minimal dependent set of M and a cocircuit is a minimal dependent set of the dual $M^\star$. A *cycle* is the symmetric difference of circuits, and a *cocycle* is the symmetric difference of cocircuits. It is well-known that a nonempty cycle is a disjoint union of circuits ([12], Theorem 9.1.2). Let $\Sigma \subseteq E(M)$. The pair (M, Σ) is called a *signed matroid*. A subset $\Gamma \subseteq E(M)$ is a *signature* of (M, Σ) if $\Sigma \triangle \Gamma$ is a cocycle of M. Observe that the symmetric difference of an odd number of signatures is another signature. For a signature Γ, the operation of replacing (M, Σ) by (M, Γ) is called *resigning*. A subset $S \subseteq E(M)$ is said to be *odd* (resp. *even*) if $|S \cap \Sigma|$ is odd (resp. even). An element $f \in E(M)$ is *odd* (resp. *even*) if $\{f\}$ is odd (resp. even). Observe that resigning a signed matroid preserves the parity of every cycle. Signed matroids are key objects as they represent binary clutters:

Proposition 2 ([8, 11], also see [3, 6])**.** *A clutter $\mathbb{F}$ is binary if, and only if, the sets of $\mathbb{F}$ are the odd circuits of a signed matroid. Moreover, assuming $\mathbb{F}$ is the clutter of odd circuits of a signed matroid, then $b(\mathbb{F})$ is precisely the set of minimal signatures.*

Take disjoint subsets $I, J \subseteq E(M)$. If I contains an odd circuit, we define $(M, \Sigma)/I \setminus J := (M/I \setminus J, \emptyset)$. Otherwise, by Proposition 2, there is a signature Σ' that is disjoint from I, and we define $(M, \Sigma)/I \setminus J := (M/I \setminus J, \Sigma' - J)$. We call $(M, \Sigma)/I \setminus J$ a *minor* of (M, Σ). Notice that minors are defined only up to resigning. We have the following relation between minors of binary clutters and minors of signed matroids:

Remark 3 (see [3])**.** *Let $\mathbb{F}$ be a binary clutter represented as the signed matroid (M, Σ). Take disjoint subsets $I, J \subseteq E(\mathbb{F})$. Then $\mathbb{F}/I \setminus J$ is represented as the signed matroid $(M, \Sigma)/I \setminus J$.*

We denote by F_7 the Fano matroid. The lines of the Fano plane $\mathbb{L}_7$ is represented as the signed matroid $(F_7, E(F_7))$. The following is an implicit result of Cornuéjols and Guenin [3], pages 349-350 (that we do not rely on):

Theorem 4. *Let $\mathbb{F}$ be an mni binary clutter represented as the signed matroid (M, Σ). If M has no F_7 minor, then $\mathbb{F} \cong \mathbb{O}_5$ or $\mathbb{F} \cong b(\mathbb{O}_5)$.*

The proof of this theorem relies on a connectivity result. To explain this result, let M be a matroid whose rank function is $r : 2^{E(M)} \to \{0, 1, 2, \ldots\}$. The *connectivity function* $\lambda_M : 2^{E(M)} \to \{0, 1, 2, \ldots\}$ is defined, for each $X \subseteq E(M)$, as $\lambda_M(X) := r(X) + r(\bar{X}) - r(E(M))$.[2] Take an integer $k \geq 1$. We say that

[2] $\bar{X} := E(M) - X$

$X \subseteq E(M)$ is *k-separating* if $\lambda_M(X) \leq k-1$. A *k-separation* is a pair $(X, \bar{X})$, where X is k-separating and $\min\{|X|, |\bar{X}|\} \geq k$. We say M is *$(k+1)$-connected* if, for each $r \in [k]$, M has no r-separation.[3] A matroid is *internally* 4*-connected* if it is 3-connected, and for every 3-separation $(X, \bar{X})$, either $|X| = 3$ or $|\bar{X}| = 3$. A key step in the proof of Theorem 4 is the following tool, that will also be essential for us:

Theorem 5 (Cornuéjols and Guenin [3], Remark 5.3, Propositions 6.1 and 7.1)**.** *Let $\mathbb{F}$ be an mni binary clutter represented as the signed matroid (M, Σ). Then M is internally 4-connected.*

For a graph G, cycle(G) will denote the cycle matroid of G, i.e. the matroid whose cycles are exactly the cycles of the graph G. Then by definition, $\mathbb{O}_5$ is represented as the signed matroid (cycle(K_5) , $E(K_5)$). Another essential tool for our work is the following result:

Theorem 6 (Guenin [5], also see Schrijver [14])**.** *Let $\mathbb{F}$ be an mni binary clutter represented as the signed matroid (M, Σ). If M is graphic, then $\mathbb{F} \cong \mathbb{O}_5$.*

The proof of Theorem 4 relies on Theorems 5 and 6, as well as Seymour's characterization of regular matroids [15]. It seems very hard to extend Theorem 4 directly to the general case of mni binary clutters, as the existence of an F_7 minor in M is not sufficient to imply the existence of an $(F_7, E(F_7))$ minor in (M, Σ). We will, however, make use of Theorems 5 and 6 in the proof of our main result, Theorem 1. In the next section we sketch the proof, and give an outline of the remainder of the paper.

2. A PROOF SKETCH OF THE MAIN RESULT

Take a matroid M. For $R \subseteq E(M)$, we write $M|R := M \setminus (E(M) - R)$. We say that $\{e_1, \ldots, e_6\} \subseteq E(M)$ is an *induced* K_4 of M if $M|\{e_1, \ldots, e_6\}$ is isomorphic to cycle(K_4).

2.1. **The clutter theoretic part.** Let $\mathbb{F}$ be an mni binary clutter with a triangle. By Proposition 2, $\mathbb{F}$ can be represented as a signed matroid. By using a seminal result of Lehman on mni clutters, we find a suitable representation of $\mathbb{F}$ to work with:

Theorem 7. *Let $\mathbb{F}$ be an mni binary clutter with a triangle. Then $\mathbb{F}$ is the clutter of odd circuits of a signed matroid $(M, E(M))$ where the following statements hold:*

a. M is internally 4-connected (Theorem 5),
b. every element in $E(M)$ is contained in exactly three triangles of M,
c. if $|E(M)| \leq 12$, then $\mathbb{F} \cong \mathbb{L}_7$ or $\mathbb{F} \cong \mathbb{O}_5$,
d. if M is graphic, then $\mathbb{F} \cong \mathbb{O}_5$ (Theorem 6),
e. if M has an induced K_4, then $\mathbb{F} \cong \mathbb{L}_7$ or $\mathbb{F} \cong \mathbb{O}_5$.

The proof of this theorem is provided in §3. After this point, we abandon the clutter $\mathbb{F}$ and work solely with the signed matroid $(M, E(M))$. We will show the following:

[3] $[k] := \{1, \ldots, k\}$

Theorem 8. *Let M be an internally 4-connected binary matroid where every element is contained in exactly three triangles. Then one of the following statements holds:*

i. $|E(M)| \leq 11$,
ii. M *is graphic,*
iii. M *has an induced* K_4*, or*
iv. the signed matroid $(M, E(M))$ *has* $(F_7, E(F_7))$ *as a minor.*

We will sketch the proof of this theorem shortly. Notice however that Theorem 1 is an immediate consequence of these two results:

Proof of Theorem 1. Let $\mathbb{F}$ be an mni binary clutter with a triangle. By Theorem 7, $\mathbb{F}$ is represented as a signed matroid $(M, E(M))$, where M is an internally 4-connected matroid and every element is contained in exactly three triangles. If either $|E(M)| \leq 12$, M is graphic, or M has an induced K_4, then by Theorem 7 (c)-(e), $\mathbb{F} \cong \mathbb{L}_7$ or $\mathbb{F} \cong \mathbb{O}_5$. Otherwise, by Theorem 8, the signed matroid $(M, E(M))$ has $(F_7, E(F_7))$ as a minor, so the mni $\mathbb{F}$ has the non-ideal $\mathbb{L}_7$ as a minor by Remark 3, implying in turn that $\mathbb{F} \cong \mathbb{L}_7$, thereby finishing the proof. □

2.2. **The matroid theoretic part.** We now sketch the proof of Theorem 8. Let M be a matroid where the following assumptions hold:

Common hypotheses

(h1) M is an internally 4-connected matroid,
(h2) every element in $E(M)$ is contained in exactly three triangles of M.

Since M is internally 4-connected, M is a simple (and cosimple) matroid. In particular, the three triangles containing an element are otherwise pairwise disjoint. Take an element $\Omega \in E(M)$. Denote the three triangles of M containing Ω by $\{\Omega, f, f'\}, \{\Omega, g, g'\}, \{\Omega, h, h'\}$. Since M is simple, M/Ω does not have a loop, and $\{f, f'\}, \{g, g'\}, \{h, h'\}$ are *the* non-trivial parallel classes of M/Ω. It follows that the simplification si(M/Ω) is obtained from M/Ω by deleting one element from each one of $\{f, f'\}, \{g, g'\}, \{h, h'\}$. If f, g, h are the elements left in si(M/Ω), we write $\Lambda(\Omega) := \{f, g, h\}$.

The proof of Theorem 8 relies on the following four propositions, as well as two theorems not done by us:

Proposition 9. *Suppose (h1)-(h2) hold and let* $\Omega \in E(M)$*. If* $\Lambda(\Omega)$ *is a cocycle of si*(M/Ω)*, then* M *has an induced* K_4.

Proof. Suppose that $\Lambda(\Omega)$ is a cocycle of si(M/e). Denote by $\{\Omega, f, f'\}, \{\Omega, g, g'\}, \{\Omega, h, h'\}$ the triangles of M containing Ω where $\Lambda(\Omega) = \{f, g, h\}$. Since $\{f, g, h\}$ is a cocycle of si(M/Ω), $D := \{f, f', g, g', h, h'\}$ is a cocycle of M/Ω and hence of M. As f is in three triangles of M, it is contained in a triangle C that is different from $\{\Omega, f, f'\}$. For D is a cocycle, $|C \cap D|$ is even, and because $f \in C \cap D$, $|C \cap D| = 2$. Moreover, $C \cap D \neq \{f, f'\}$, for otherwise $C \triangle \{f, f', \Omega\}$ would be a cycle of cardinality two, which cannot be the case as M is simple. Hence, we may assume that $C \cap D = \{f, g\}$ or $C \cap D = \{f, g'\}$. In either cases, $C \cup \{\Omega, f, f', g, g'\}$ is an induced K_4 of M, as required. □

We require a few preliminaries to prove the next proposition. A *graft* is a pair (G, T), where G is a graph and $T \subseteq V(G)$ is of even cardinality. Vertices in T are called *terminals*. Take a subset $J \subseteq E(G)$. Denote by $\text{odd}(J) \subseteq V(G)$ the vertices incident with an odd number of non-loop edges in J. If $\text{odd}(J) = T$, then we call J a *T-join*. Start with the vertex-edge incidence matrix of G, and add the vertex-incidence vector of T as a column; call this matrix A. Let M be the (binary) matroid whose binary representation is A, and denote by t the element of M corresponding to column T. Then $C \subseteq E(M)$ is a cycle of M if, and only if, one of the following holds:

- $t \notin C$ and C is a cycle of G,
- $t \in C$ and $C - t$ is a T-join of G.

We call M the *graft matroid* of (G, T). By convention, t will always be the element of M corresponding to the terminals T. Notice that if $|T| \leq 2$, then the graft matroid of (G, T) is graphic. The next folklore remark states that graft matroids are precisely those matroids that are one deletion away from being graphic (see for instance [12], Lemma 10.3.8):

Remark 10. *Take a binary matroid M and an element $t \in E(M)$ such that $M \setminus t = cycle(G)$, for some graph G. If C is a cycle of M containing t, then M is the graft matroid of the graft $(G, odd(C - t))$.*

We are now ready for the next proposition:

Proposition 11. *Suppose (h1)-(h2) hold and let $\Omega \in E(M)$. If $M \setminus \Omega$ is graphic, then M is graphic or has an induced K_4.*

Proof. Suppose $M \setminus \Omega$ is graphic. By Remark 10, there is a graft (G, T) whose graft matroid is M, where $t = \Omega$. If $|T| \leq 2$, then M is graphic, so we are done. Otherwise, $|T| \geq 4$. Denote the three triangles of M containing Ω by $\{\Omega, f, f'\}, \{\Omega, g, g'\}, \{\Omega, h, h'\}$. Then $\{f, f'\}, \{g, g'\}$ and $\{h, h'\}$ are T-joins of G. Since M is simple, we see that G does not have parallel edges. As a result, $|T| = 4$ and $\{f, f', g, g', h, h'\}$ is an induced K_4 of M, as required. □

Proposition 12. *Suppose (h1)-(h2) hold and let $\Omega \in E(M)$. If $\Lambda(\Omega)$ is contained in a circuit of $si(M/\Omega)$, then either M has an induced K_4 or $(M, E(M))$ has $(F_7, E(F_7))$ as a minor.*

We prove this proposition in §5.4.

Proposition 13. *Suppose (h1)-(h2) hold and let e_1, e_2, e_3, e_4 be distinct elements of M such that, for every $i \in [4]$, $si(M/e_i)$ is internally 4-connected and is the cycle matroid of a graph where the three edges of $\Lambda(e_i)$ are incident to the same vertex. Then either $|E(M)| \leq 11$, or there is an element $\Omega \in E(M)$ such that $M \setminus \Omega$ is graphic.*

This proposition is proved in §7. We will also need the following result of Seymour [19] that characterizes, under appropriate connectivity conditions, when three distinct elements of a matroid are contained in a circuit:

Theorem 14. *Let M be an internally 4-connected binary matroid, and let f, g, h be distinct elements. Then one of the following statements holds:*

a. *$\{f, g, h\}$ is contained in a circuit of M,*
b. *$\{f, g, h\}$ is a cocycle of M, or*
c. *M is the cycle matroid of a graph where edges f, g, h are incident to the same vertex.*

The following result of Chun and Oxley [2] on internally 4-connected matroids is the last needed ingredient:

Theorem 15. *Let M be an internally 4-connected binary matroid where every element is in exactly three triangles. Then there exist distinct elements $e_1, e_2, e_3, e_4 \in E(M)$ such that, for each $j \in [4]$, si(M/e_j) is internally 4-connected.*

We are now ready to prove Theorem 8:

Proof of Theorem 8. Suppose (h1)-(h2) hold. By Theorem 15, there exist distinct elements e_1, e_2, e_3, e_4 of M such that, for each $j \in [4]$, $\mathrm{si}(M/e_j)$ is internally 4-connected. For $j \in [4]$,

- if $\Lambda(e_j)$ is contained in a circuit of $\mathrm{si}(M/e_j)$, then by Proposition 12, either M has an induced K_4 and so (iii) holds, or $(M, E(M))$ has $(F_7, E(F_7))$ as a minor and so (iv) holds,
- if $\Lambda(e_j)$ is a cocycle of $\mathrm{si}(M/e_j)$, then by Proposition 9, M has an induced K_4, so (iii) holds.

Otherwise, it follows from Theorem 14 that, for each $j \in [4]$, $\mathrm{si}(M/e_j)$ is the cycle matroid of a graph where the three edges in $\Lambda(e_j)$ are incident to the same vertex. By Proposition 13, either $|E(M)| \leq 11$ and so (i) holds, or there is an element $\Omega \in E(M)$ such that $M \setminus \Omega$ is graphic. By Proposition 11, either M is graphic and so (ii) holds, or M has an induced K_4 and so (iii) holds. In all cases, one of (i)-(iv) holds, and so we are done. □

2.3. **Outline of the paper.** In §3 we review Lehman's theorem on mni clutters and prove Theorem 7. In §4 we introduce *signed grafts* and present two instances that have $(F_7, E(F_7))$ as a minor. In §5 we leverage these results to prove Proposition 12. In §6 we introduce *even cycle matroids* and prove several relevant results, which in turn lead to a proof of Proposition 13 in §7.

3. Lehman and Theorem 7

Let $\mathbb{F}$ be a clutter. We denote by $M(\mathbb{F})$ the $0,1$ matrix whose columns are indexed by $E(\mathbb{F})$ and whose rows are the incidence vectors of the sets in $\mathbb{F}$. The clutter of the minimum cardinality sets in $\mathbb{F}$ is denoted by $\bar{\mathbb{F}}$. For an integer $k \geq 1$, a $0,1$ matrix is *k-regular* if each row and each column has exactly k ones. Lehman [9] (see Seymour [16]) proved a structural result on mni clutters; we only need the binary version of his result:

Theorem 16. *Let $\mathbb{F}$ be an mni binary clutter. Then $\mathbb{K} := b(\mathbb{F})$ is also mni, and the following statements hold:*

a. *$M(\bar{\mathbb{F}})$ and $M(\bar{\mathbb{K}})$ are square matrices,*
b. *for some integers $r \geq 3$ and $s \geq 3$, $M(\bar{\mathbb{F}})$ is r-regular and $M(\bar{\mathbb{K}})$ is s-regular,*

c. *for* $n := |E(\mathbb{F})|$, $rs - n$ *is an even integer such that* $2 \le rs - n \le \min\{r-1, s-1\}$, *and*

d. *after possibly rearranging the rows of* $M(\bar{\mathbb{K}})$, *we have*

$$M(\bar{\mathbb{F}})M(\bar{\mathbb{K}})^\top = J + (rs-n)I.$$

Here, J *is the all-ones matrix and* I *is the identity matrix.*

Notice that if $\mathbb{F}$ is an mni binary clutter with a triangle, then $r = 3$ and $3s - n = 2$. The following is therefore an easy consequence (see for instance [10]):

Remark 17. *Let* $\mathbb{F}$ *be an mni binary clutter with a triangle, and let* $\mathbb{K} := b(\mathbb{F})$. *Denote by* s *the minimum cardinality of a set in* $\mathbb{K}$. *Then,*

a. *if* $s = 3$ *then* $\mathbb{F} \cong \mathbb{L}_7$, *and*

b. *if* $s = 4$ *then* $\mathbb{F} \cong \mathbb{O}_5$.

Bridges and Ryser [1] showed that the two matrices satisfying the equation in (d) commute:

Theorem 18. *Take square* $0, 1$ *matrices* A, B *such that for some integer* $d \ge 1$, $AB = J + dI$. *Then* $AB = BA$.

We are now ready to prove Theorem 7:

Proof of Theorem 7. Let $\mathbb{F}$ be an mni binary clutter with a triangle, and set $n := |E(\mathbb{F})|$. Let $\mathbb{K} := b(\mathbb{F})$ and denote by s the minimum cardinality of a set in $\mathbb{K}$. By Theorem 16, after possibly rearranging the rows of $M(\bar{\mathbb{K}})$,

$$(\star) \qquad r = 3 \quad \text{and} \quad s \ge 3 \quad \text{and} \quad 3s - n = 2 \quad \text{and} \quad M(\bar{\mathbb{F}})M(\bar{\mathbb{K}})^\top = J + 2I.$$

Note further that $\bar{\mathbb{F}}$ is precisely the clutter of the triangles of $\mathbb{F}$, and since $M(\bar{\mathbb{F}})$ is 3-regular, every element of $E(\mathbb{F})$ is contained in exactly 3 triangles of $\mathbb{F}$. Label the rows of $M(\bar{\mathbb{F}})$ as $S_1, \ldots, S_n \in \bar{\mathbb{F}}$, and the rows of $M(\bar{\mathbb{K}})$ as $R_1, \ldots, R_n \in \bar{\mathbb{K}}$. Then the last equation implies, for all $i, j \in [n]$, that

$$|S_i \cap R_j| = \begin{cases} 3 & \text{if } i = j, \\ 1 & \text{if } i \ne j. \end{cases}$$

For each $i \in [n]$, we say that S_i and R_i are *mates* of one another. Thus, a triangle of $\mathbb{F}$ is contained in its mate, and it intersects all the other triangle mates exactly once.

1. *Take an element* $e \in E(\mathbb{F})$, *denote by* S, S', S'' *the triangles of* $\mathbb{F}$ *containing* e, *and by* R, R', R'' *their respective mates in* $\bar{\mathbb{K}}$. *Then*

- $R \cap R' = R' \cap R'' = R'' \cap R = \{e\}$,
- $R \cup R' \cup R'' = E(\mathbb{F})$, *and*
- $S \cap S' = S' \cap S'' = S'' \cap S = \{e\}$.

Subproof. It follows from $(\star)$ and Theorem 18 that $M(\bar{\mathbb{K}})^\top M(\bar{\mathbb{F}}) = J + 2I$. Denote by c_e the column of $M(\bar{\mathbb{F}})$ corresponding to e, and for each $f \in E(\mathbb{F})$, denote by c'_f the column of $M(\bar{\mathbb{K}})$ corresponding to f. Then the matrix equation implies that $c_e^\top c'_e = 3$ and, for each $f \in E(\mathbb{F}) - e$, that $c_e^\top c'_f = 1$; the first and second lines follow. Since $S \subseteq R$, $S' \subseteq R'$ and $S'' \subseteq R''$, the third line follows. $\Diamond$

Since $\mathbb{F}$ is a binary clutter, we get from Proposition 2 that $\mathbb{F}$ is the clutter of odd circuits of a signed matroid (M, Σ).

2. *$E(M)$ is a signature of (M, Σ).*

Subproof. Take $e \in E(\mathbb{F})$ and let R, R', R'' be the mates of the triangles of $\mathbb{F}$ containing e. Since R, R', R'' belong to $b(\mathbb{F})$, they are signatures of (M, Σ) by Proposition 2. So their symmetric difference $R \triangle R' \triangle R''$ is also a signature. However, (1) implies that $R \triangle R' \triangle R'' = E(M)$, so $E(M)$ is a signature. ♢

Thus, $\mathbb{F}$ is the clutter of odd circuits of the signed matroid $(M, E(M))$. It follows from Theorem 5 that M is internally 4-connected, so (a) holds.

3. *Every element of $E(M)$ is contained in exactly 3 triangles of M, so (b) holds.*

Subproof. Since $\mathbb{F}$ is the clutter of odd circuits of $(M, E(M))$, the triangles of $\mathbb{F}$ are precisely the triangles of M. Since every element of $E(\mathbb{F})$ is contained in exactly 3 triangles of $\mathbb{F}$, the claim follows. ♢

4. *If $|E(M)| \leq 12$, then $\mathbb{F} \cong \mathbb{L}_7$ or $\mathbb{F} \cong \mathbb{O}_5$, so (c) holds.*

Subproof. By $(\star)$, $3s - 2 = n = |E(\mathbb{F})| = |E(M)| \leq 12$ and $s \geq 3$, so $s \in \{3, 4\}$ and by Remark 17, we get that $\mathbb{F} \cong \mathbb{L}_7$ or $\mathbb{F} \cong \mathbb{O}_5$. ♢

It follows from Theorem 6 that if M is graphic, then $\mathbb{F} \cong \mathbb{O}_5$, so (d) holds. It remains to prove (e). To this end, assume that M has an induced K_4, that is, there are elements $e_1, \ldots, e_6 \in E(M)$ such that $M|\{e_1, \ldots, e_6\} \cong$ cycle(K_4). As the triangles of M are precisely the triangles of $\mathbb{F}$, we may assume that S_1, S_2, S_3, S_4 are the four triangles of $M|\{e_1, \ldots, e_6\}$.

5. *For all distinct $i, j \in [4]$, $R_i \cap R_j \subseteq \{e_1, \ldots, e_6\}$.*

Subproof. As S_i, S_j are distinct triangles of K_4, there is an $e \in \{e_1, \ldots, e_6\}$ such that $S_i \cap S_j = \{e\}$. It now follows from (1) that $R_i \cap R_j = \{e\} \subseteq \{e_1, \ldots, e_6\}$. ♢

6. *For all $i \in [4]$, $R_i \cap \{e_1, \ldots, e_6\} = S_i$ and $|R_i - \{e_1, \ldots, e_6\}| = s - 3$.*

Subproof. Since R_i is the mate of S_i, we have $S_i \subseteq R_i$. As R_i intersects every other triangle exactly once, and $|S_i \cap S_j| = 1$ for each $j \in [4] - i$, we get that $R_i \cap \{e_1, \ldots, e_6\} = S_i$. ♢

Putting (5) and (6) together, we get that $|E(M)| \geq 6 + 4(s - 3)$. From $(\star)$ we have that $s \geq 3$, and also that $|E(M)| = |E(\mathbb{F})| = n = 3s - 2$, so $3s - 2 \geq 6 + 4(s - 3)$, implying in turn that $s \in \{3, 4\}$. It now follows Remark 17 that $\mathbb{F} \cong \mathbb{L}_7$ or $\mathbb{F} \cong \mathbb{O}_5$, thereby proving (e). This finishes the proof of Theorem 7. □

4. Quadrums and Trifolds

4.1. **Representations of the Fano matroid.** A *plain quadrum* is the graft $(K_4, V(K_4))$. A *plain trifold* is the graft where, the graph has vertex set [5] and edges $\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}$, and the terminals are $\{2, 3, 4, 5\}$. Drawings of the plain quadrum and the plain trifold are given in Figure 1.

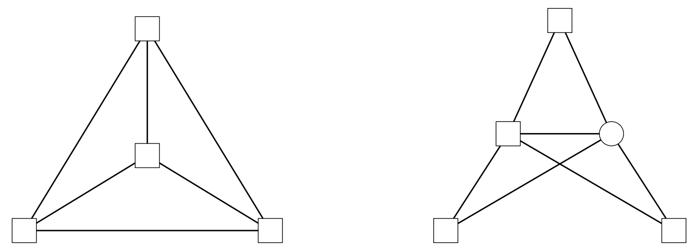


Figure 1. Left: plain quadrum, right: plain trifold. Square vertices are terminals.

Remark 19. *Let (G, T) be a graft, and N its graft matroid. Then the following statements hold:*

a. if (G, T) is a plain quadrum, then $N \cong F_7$,

b. if (G, T) is a plain trifold, then $N/t \cong F_7$.

Proof. Notice that a matroid is determined by the set of its circuits. **(a)** Consider Figure 2 (a). We assign t and each edge of the plain quadrum to an element of F_7. It now suffices to observe that the circuits of N correspond to the circuits of F_7, i.e. to the lines and the line complements of the Fano plane. **(b)** Consider Figure 2 (b). We assign each edge of the plain trifold to an element of F_7. Observe that the circuits of N/t, which are the circuits and T-joins of G, correspond to the circuits of of F_7. □

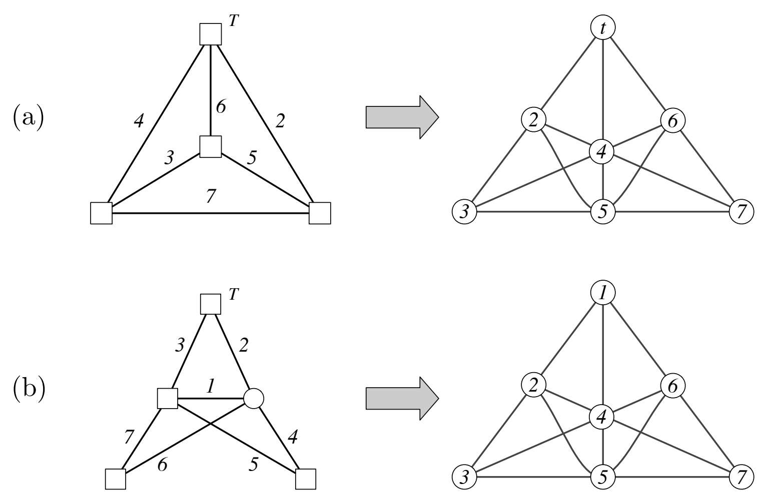


Figure 2. The Fano matroid in disguise

4.2. **Signed grafts: quadrums and trifolds.** A *signed graft* is a triple (G, T, Γ), where (G, T) is a graft and $\Gamma \subseteq E(G) \cup \{t\}$. Note that we assign parity to each edge as well as to the set of terminals.

A *quadrum* is the signed graft (G, T, Γ) where (G, T) is a plain quadrum and $\Gamma = E(G) \cup \{t\}$. A *super quadrum* is the signed graft displayed in Figure 3 (a) which is obtained as follows: start with a plain quadrum, take a set S of four edges that contain a triangle, the element t and the two edges outside S can have either parities, and replace each edge of S by a pair of parallel edges of distinct parities.

A *trifold* is the signed graft (G,T,Γ) where (G,T) is a plain trifold and $\Gamma = E(G)$. A *super trifold* is the signed graft displayed in Figure 3 (b) which is obtained as follows: start with a plain trifold, take two triangles and the two edges S disjoint from them, the edges outside S are odd, replace each edge of S by a pair of parallel edges of distinct parities, and element t can have either parities.

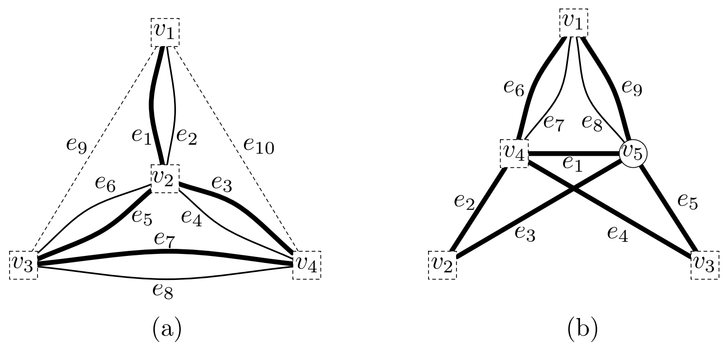


Figure 3. (a) Super quadrum, (b) super trifold. Square vertices are terminals. Bold edges are odd. Thin edges are even. Dashed edges can be either odd or even. In (a) and (b), t can be odd or even.

4.3. **Segway to** $(F_7, E(F_7))$**.** Let G be a graph. For a vertex $v \in V(G)$, we denote by $\delta_G(v)$ the set of non-loop edges of G that are incident with v. Take a signed graft (G,T,Γ) and a terminal $v \in T$. Let $B := \delta_G(v) \cup \{t\}$. We say that $(G,T,\Gamma \triangle B)$ is obtained from (G,T,Γ) by *resigning on the terminal* v.

Remark 20. *Let (G,T,Γ) be a signed graft, and N the graft matroid of the graft. If (G,T,Γ') is obtained from (G,T,Γ) by resigning on a terminal, then Γ' is a signature of the signed matroid (N,Γ).*

Proof. It suffices to show that, for each terminal $v \in T$, the set $B := \delta_G(v) \cup \{t\}$ is a cocycle of N. To this end, let C be a cycle of N. If $t \notin C$, then C is a cycle of G, and so $|C \cap B| = |C \cap \delta_G(v)|$ even. Otherwise, $t \in C$ and $C - t$ is a T-join of G. Since $v \in T$, $|(C-t) \cap \delta_G(v)|$ must be odd, implying in turn that $|C \cap B|$ is even. In both cases, for every cycle C of N, $|C \cap B|$ is even, which means that B is a cocycle of N. □

Proposition 21. *Let (G,T,Γ) be a signed graft, and N the graft matroid of the graft. If (G,T,Γ) is a quadrum, a super quadrum, a trifold, or a super trifold, then (N,Γ) has $(F_7, E(F_7))$ as a minor.*

Proof. **Case 1:** (G,T,Γ) is a quadrum. By definition, (G,T) is a plain quadrum and $\Gamma = E(N)$. Remark 19 states that $N \cong F_7$. Hence, $(N,\Gamma) \cong (F_7, E(F_7))$. **Case 2:** (G,T,Γ) is a super quadrum. Label the vertices and edges of G as in Figure 3 (a). After possibly resigning on terminals v_2, v_3, v_4, we may assume by Remark 20 that $\Gamma = \{e_1, e_3, e_5, e_7, e_9, e_{10}, t\}$. Let $(N',\Gamma) := (N,\Gamma) \setminus \{e_2,e_4,e_6,e_8\}$ and $G' := G \setminus \{e_2,e_4,e_6,e_8\}$. Then N' is the graft matroid of (G',T), and (G',T,Γ) is a quadrum. It therefore follows from Case 1 that $(N,\Gamma) \setminus \{e_2,e_4,e_6,e_8\} = (N',\Gamma) \cong (F_7, E(F_7))$. **Case 3:** (G,T,Γ) is a trifold. By definition, (G,T) is a plain trifold and $\Gamma = E(N) - t$. Remark 19 states that $N/t \cong F_7$, implying in turn that $(N,\Gamma)/t \cong (F_7, E(F_7))$. **Case 4:** (G,T,Γ) is a super trifold. Label the vertices and edges of G as in Figure 3 (b). After possibly resigning on terminal v_1, we may assume by Remark 20 that $\Gamma = \{e_1, e_2, e_3, e_4, e_5, e_6, e_9\}$. Let $(N', \Gamma) := (N, \Gamma) \setminus \{e_7, e_8\}$ and $G' := G \setminus \{e_7, e_8\}$. Then N' is the graft matroid of (G', T), and (G', T, Γ) is a trifold. It therefore follows from Case 3 that $(N, \Gamma) \setminus \{e_7, e_8\}/t = (N', \Gamma)/t \cong (F_7, E(F_7))$. $\square$

5. Proposition 12

5.1. **Starting the proof.** Suppose (h1)-(h2) hold, that is, M is an internally 4-connected matroid where every element is in exactly three triangles. Let $\Omega \in E(M)$. We would like to show that if $\Lambda(\Omega)$ is contained in a circuit of $\mathrm{si}(M/\Omega)$, then either M has an induced K_4 or $(M, E(M))$ has an $(F_7, E(F_7))$ minor. Let us make the following assumptions:

Further hypotheses

- (h3) $\Omega \in E(M)$ is contained in the triangles $\{\Omega, f, f'\}$, $\{\Omega, g, g'\}$, $\{\Omega, h, h'\}$ where $\Lambda(\Omega) = \{f, g, h\}$,
- (h4) $M_\Omega := M/\Omega \setminus \{f', g', h'\}$ and $(M_\Omega, \Sigma_\Omega) := (M, E(M))/\Omega \setminus \{f', g', h'\}$,
- (h5) C is a circuit in M_Ω of minimum cardinality that contains $\{f, g, h\}$,
- (h6) M does not have an induced K_4.

Note that $M_\Omega = \mathrm{si}(M/\Omega)$. We leave the following as an easy exercise for the reader:

Remark 22. *Take a binary matroid N, an element $e \in E(N)$, and a subset $D \subseteq E(N)$. Then the following statements hold:*

- *a. if D is a circuit of N/e, then exactly one of $D, D \cup \{e\}$ is a circuit of N,*
- *b. if D is a cycle of N/e, then at least one of $D, D \cup \{e\}$ is a cycle of N,*
- *c. if D is a cycle of N and $e \in D$, then $D - e$ is a cycle of N/e, and*
- *d. if D is a cycle of N and $e \notin D$, then D is a cycle of N/e.*

A key milestone in the proof is the proposition below:

Proposition 23. *Suppose (h1)-(h6) hold. Let D be a circuit of M_Ω containing $\{f, g, h\}$. Then,*

- *a. the elements of $D - \{f, g, h\}$ are in series in $M|D \cup \{f', g', h', \Omega\}$,*
- *b. $D - \{f, g, h\} \neq \emptyset$, and for each $t \in D - \{f, g, h\}$, $M|\big(D \cup \{f', g', h', \Omega\}\big)/(D - \{f, g, h, t\})$ is the graft matroid of a plain trifold,[4] and*
- *c. if $|D|$ is odd, then $(M, E(M))$ has an $(F_7, E(F_7))$ minor.*

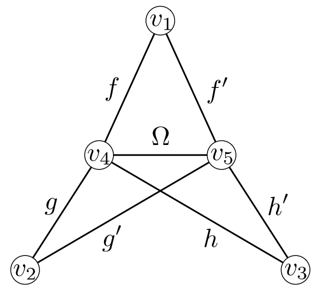
Proof. Let $M' := M|D \cup \{f', g', h', \Omega\}$.

(a) Note that M'/Ω consists of the circuit D together with edges f', g', h'. Since $\{f, f'\}, \{g, g'\}, \{h, h'\}$ are parallel classes of M'/Ω, it follows that the elements of $D - \{f, g, h\}$ are in series in M'/Ω, so they are also in series in M'. **(b)** If $\{f, g, h\}$ or $\{f, g, h, \Omega\}$ is a circuit of M, then $\{\Omega, f, f', g, g', h\}$ would be an induced K_4 of M, which cannot occur by (h6). Hence, neither $\{f, g, h\}$ nor $\{f, g, h, \Omega\}$ is a circuit of M – this has two consequences. (1) Since one of $D, D \cup \{\Omega\}$ is a circuit of M by Remark 22 (a), we get that $D \neq \{f, g, h\}$.

[4] $M|I/J$ is short-hand notation for $(M|I)/J$.

(2) The set $\{f,g,h,\Omega\}$ is independent in the matroid M, and this is in turn implies that $M|\{f,f',g,g',h,h',\Omega\}$ is the cycle matroid of the graph G displayed below on vertices $\{v_1,v_2,v_3,v_4,v_5\}$ and edges

$$\Omega=\{v_4,v_5\},\ f=\{v_1,v_4\},\ f'=\{v_1,v_5\},\ g=\{v_2,v_4\},\ g'=\{v_2,v_5\},\ h=\{v_3,v_4\},\ h'=\{v_3,v_5\}.$$



Let $t\in D-\{f,g,h\}$ and

$$N:=M'/(D-\{f,g,h,t\}).$$

By (a), the elements of $D-\{f,g,h\}$ are in series in M', so $M'/(D-\{f,g,h,t\})\setminus t=M'\setminus(D-\{f,g,h\})$, implying in turn that $N\setminus t=M|\{f,f',g,g',h,h',\Omega\}=\text{cycle}(G)$. It therefore follows from Remark 10 that, for some $T\subseteq V(G)$ of even cardinality, N is the graft matroid of the graft (G,T). Since one of $D,D\cup\{\Omega\}$ is a circuit of M by Remark 22 (a), it follows that one of $\{f,g,h,t\},\{f,g,h,t,\Omega\}$ is a circuit of N, implying in turn that one of $\{f,g,h\},\{f,g,h,\Omega\}$ is a T-join of G. This means that $T=\{v_1,v_2,v_3,v_4\}$ or $T=\{v_1,v_2,v_3,v_5\}$. Either way, we see that (G,T) is a plain trifold. **(c)** Assume that $|D|$ is odd. Let $\Gamma:=\{\Omega,f,f',g,g',h,h'\}$. Notice that (G,T,Γ) is a trifold. Thus, by Proposition 21, (N,Γ) has an $(F_7,E(F_7))$ minor. It therefore suffices to show that (N,Γ) is a minor of $(M',E(M'))$, which itself is a minor of $(M,E(M))$. Since $|D|$ is odd, $D-\{f,g,h\}$ has an even number of elements, all of which are in series in M' by (a), so $D-\{f,g,h\}$ is a cocycle of M'. As a result, $E(M')\triangle(D-\{f,g,h\})=\Gamma$ is a signature of $(M',E(M'))$. However, $(M',\Gamma)/(D-\{f,g,h,t\})=(N,\Gamma)$, so (N,Γ) is a minor of $(M',E(M'))$, as required. □

We may therefore make the following assumptions:

Further hypotheses

(h7) every circuit of M_Ω containing $\{f,g,h\}$ has even cardinality.

In particular, $|C|$ is even and so $C-\{f,g,h\}\neq\emptyset$. Let S be a triangle of M_Ω containing an element of $C-\{f,g,h\}$. We say that S is *f-splitting* if either $S\cap\{f,g,h\}=\{f\}$ or the following statements hold:

- $|S\cap C|=1$, and
- $S\triangle C$ is the union of two disjoint circuits of M_Ω, one of which contains f and the other contains g,h.

Similarly, we have *g-splitting* and *h-splitting* triangles.

Corollary 24. *Suppose (h1)-(h7) hold. Then every triangle of M_Ω containing an element of $C-\{f,g,h\}$ is a splitting triangle.*

Proof. Take an element $e \in C - \{f, g, h\}$ and a triangle S of M_Ω such that $e \in S$. Notice that $|S \cap \{f, g, h\}| \leq 1$. So if $S \cap \{f, g, h\} \neq \emptyset$, then S is a splitting triangle. We may therefore assume that $S \cap \{f, g, h\} = \emptyset$. Clearly, $1 \leq |S \cap C| \leq 2$. Note that $|S \cap C| = 1$; for if not, then $S \triangle C$ would be an odd-length circuit of M_Ω containing $\{f, g, h\}$, which cannot occur by (h7). Consider now the odd-length cycle $S \triangle C$, which is either a circuit or the disjoint union of two circuits. However, it follows from (h7) that $S \triangle C$ is the union of two disjoint circuits, both of which contain elements from $\{f, g, h\}$. This implies that R is a splitting triangle. □

The rest of this section is organized as follows: we will show that

- unless $(M, E(M))$ has an $(F_7, E(F_7))$ minor, every element of $C - \{f, g, h\}$ is in three otherwise disjoint triangles of M_Ω, one of which is f-splitting, the second one is g-splitting, and the third one is h-splitting (§5.3),
- the circuit C, together with its splitting triangles, gives rise to a so-called Type I or a Type II configuration in $(M_\Omega, \Sigma_\Omega)$ (§5.4),
- a Type I configuration gives a super trifold minor in $(M, E(M))$, and a Type II configuration gives a super quadrum minor in $(M, E(M))$ (§5.2),

and by Proposition 21, the last step leads to an $(F_7, E(F_7))$ minor, thereby finishing the proof of Proposition 12.

5.2. **Type I and Type II configurations.** In M_Ω, take an element $p \in E(M_\Omega) - C$ that is spanned by C. Then $C \cup \{p\}$ contains exactly three circuits, one of which is C, the other two contain p and their symmetric difference is C; notice that the other two circuits either separate f, g, h or not. We say that p is *f-splitting* if there is a circuit in $C \cup \{p\}$ that contains f and none of g, h. Observe that if S is an f-splitting triangle, then each element of $S - C$ is f-splitting. Similarly, we have *g-splitting* and *h-splitting* elements. If p is e-splitting, for some $e \in \{f, g, h\}$, we denote by $\Theta(p)$ the circuit contained in $C \cup \{p\}$ such that $\Theta(p) \cap \{f, g, h\} = \{e\}$.

In this section, we identify two configurations of splitting elements and show that their presence implies the existence of a super trifold or super quadrum minor in $(M, E(M))$.

We say $\{p_1, p_2\} \subseteq E(M_\Omega)$ is a *Type I configuration* if the following statements hold:

- p_1 and p_2 are e-splitting, for some $e \in \{f, g, h\}$,
- $C - \big(\Theta(p_1) \cup \Theta(p_2) \cup \{f, g, h\}\big) \neq \emptyset$,
- $\Theta(p_1) \triangle \Theta(p_2)$ is an odd cycle of $(M_\Omega, \Sigma_\Omega)$,
- $|\Theta(p_1)|$ and $|\Theta(p_2)|$ are odd.

We will show that a Type I configuration leads to a super trifold in $(M, E(M))$. To prove this, however, we need an ingredient. Recall that by Remark 22 (a), if D is a circuit of M/Ω, then exactly one of $D, D \cup \{\Omega\}$ is a circuit of M; the following proposition characterizes when D is the circuit in M:

Remark 25. *Suppose (h1)-(h7) hold. Let D be a circuit of M/Ω. Then D is a circuit of M if, and only if, the parity of $|D|$ is equal to the parity of D in $(M, E(M))/\Omega$. In particular, if D is a circuit of M_Ω, then the following statements are equivalent:*

i. D is a circuit of M,

ii. $|D|$ *and* $|D \cap \Sigma_\Omega|$ *have the same parity.*

Proof. Let D be a circuit of M/Ω. Assume that D is a circuit of M. Then the parity of $|D|$ is equal to the parity of D in $(M, E(M))$, which is equal to the parity of D in $(M, E(M))/\Omega$. Conversely, assume that the parity of $|D|$ is equal to the parity of D in $(M, E(M))/\Omega$. Suppose, for a contradiction, that D is not a circuit of M. By Remark 22 (a), $D \cup \{\Omega\}$ is a circuit of M, and moreover, the parity of $D \cup \{\Omega\}$ in $(M, E(M))$ is equal to the parity of D in $(M, E(M))/\Omega$, which by assumption is equal to the parity of $|D|$. However, the parity of $D \cup \{\Omega\}$ in $(M, E(M))$ is equal to the parity of $|D \cup \{\Omega\}| = |D| + 1$, a contradiction. □

Proposition 26. *Suppose (h1)-(h7) hold. If there is a Type I configuration, then* $(M, E(M))$ *has* $(F_7, E(F_7))$ *as a minor.*

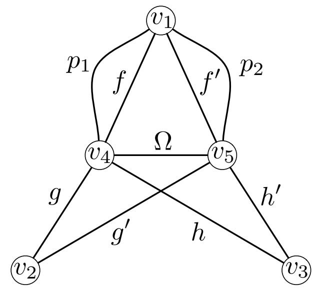
Proof. Assume that there is a Type I configuration $\{p_1, p_2\}$. After possibly interchanging the roles of f, g, h, we may assume that p_1, p_2 are f-splitting, and after possibly interchanging the roles of p_1, p_2, we may assume that $\Theta(p_1)$ is an odd circuit and $\Theta(p_2)$ an even circuit of $(M_\Omega, \Sigma_\Omega)$. Since $|\Theta(p_1)\,|$ and $|\Theta(p_2)\,|$ are odd, it follows from Remark 25 that $\Theta(p_1)$ is an odd circuit of $(M, E(M))$, and together with Remark 22 (a), that $\Theta(p_2) \cup \{\Omega\}$ is an even circuit of $(M, E(M))$. Now take an element $t \in C - \big(\Theta(p_1) \cup \Theta(p_2) \cup \{f, g, h\}\big)$. Consider the following minor of $(M, E(M))$:

$$(N, \Gamma) := (M, E(M))|(C \cup \{f', g', h', \Omega, p_1, p_2\})/(C - \{f, g, h, t\}).$$

We will show that (N, Γ) corresponds to a super trifold. By Proposition 23 (b), $N \setminus \{p_1, p_2\}$ is the graft matroid of a plain trifold (G', T), where $V(G') = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G')$ consists of

$$\Omega = \{v_4, v_5\},\ f = \{v_1, v_4\},\ f' = \{v_1, v_5\},\ g = \{v_2, v_4\},\ g' = \{v_2, v_5\},\ h = \{v_3, v_4\},\ h' = \{v_3, v_5\},$$

and either $T = \{v_1, v_2, v_3, v_4\}$ or $T = \{v_1, v_2, v_3, v_5\}$. (See below for an illustration.) Since the triangles $\{\Omega, f, f'\}, \{\Omega, g, g'\}, \{\Omega, h, h'\}$ are odd in the signed matroid $(M, E(M))$, they are odd also in the minor $(N, \Gamma) \setminus \{p_1, p_2\} = \big(N \setminus \{p_1, p_2\}, \Gamma - \{p_1, p_2\}\big)$. We may therefore assume that $\{\Omega, f, f', g, g', h, h'\} \subseteq \Gamma - \{p_1, p_2\}$. Notice that we do not know whether or not t belongs to $\Gamma - \{p_1, p_2\}$. Since $\Theta(p_1)$ is a circuit of M containing $\{f, p_1\}$ and all of its other edges belong to $C - \{f, g, h, t\}$, it follows that $\{f, p_1\}$ is a circuit of N. Similarly, since $\Theta(p_2) \cup \{\Omega\}$ is a circuit of M containing $\{f, p_2, \Omega\}$ and all of its other edges belong to $C - \{f, g, h, t\}$, we get that $\{f, p_2, \Omega\}$ is a triangle of N, which in turn implies that $\{f', p_2\}$ is a circuit of N. As a consequence, N is the graft matroid of the graft (G, T) obtained from (G', T) after adding edge p_1 parallel to f, and edge p_2 parallel to f'. Since $\Theta(p_1)$ is an odd circuit of $(M, E(M))$, we get that $\{f, p_1\}$ is an odd circuit of (N, Γ), so $p_1 \notin \Gamma$. Similarly, as $\Theta(p_2) \cup \{\Omega\}$ is an even circuit of $(M, E(M))$, we get that $\{f, p_2, \Omega\}$ is an even triangle of (N, Γ), and as $\{f, f', \Omega\}$ is an odd triangle, we have that $\{f', p_2\}$ is also an odd circuit of (N, Γ). Hence, $p_2 \notin \Gamma$. Therefore, the signed graft (G, T, Γ) is a super trifold.



It follows from Proposition 21 that (N,Γ), and therefore $(M,E(M))$, has an $(F_7,E(F_7))$ minor, as desired. $\square$

We say $\{p_1,p_1',p_2,p_3\}\subseteq E(M_\Omega)$ is a *Type II configuration* if the following statements hold:

- p_1 and p_1' are e_1-splitting, p_2 is e_2-splitting, and p_3 is e_3-splitting, for a permutation e_1,e_2,e_3 of f,g,h,
- $\Theta(p_1)\cap\Theta(p_1')\cap\Theta(p_2)\cap\Theta(p_3)\neq\emptyset$,
- $\Theta(p_1)\triangle\Theta(p_1')$ is an odd cycle of (M_Ω,Σ_Ω).

We will show that a Type II configuration leads to a super quadrum in $(M,E(M))/\Omega$, and therefore, in $(M,E(M))$:

Proposition 27. *Suppose (h1)-(h7) hold. If there is a Type II configuration, then $(M,E(M))$ has $(F_7,E(F_7))$ as a minor.*

Proof. Assume that there is a Type II configuration $\{p_1,p_1',p_2,p_3\}$. By symmetry, we may assume that p_1,p_1' are f-splitting, p_2 is g-splitting, and p_3 is h-splitting. Take an element $t\in\Theta(p_1)\cap\Theta(p_1')\cap\Theta(p_2)\cap\Theta(p_3)$. Observe that $(M',\Sigma'):=(M,E(M))/\Omega$ is obtained from (M_Ω,Σ_Ω) after adding edges f',g',h' parallel of different parity to f,g,h, respectively. Consider now the minor

$$(N,\Gamma):=(M',\Sigma')|\big(C\cup\{p_1,p_1',p_2,p_3\}\big)/\big(C-\{f,g,h,t\}\big).$$

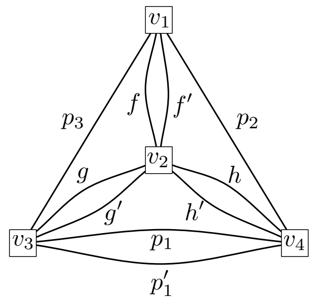
We will show that (N,Γ) corresponds to a super quadrum.

Since C is a circuit of M', it follows that $\{f,g,h,t\}$ is a circuit of N. Start with the graft (G'',T) on vertices $\{v_1,v_2,v_3,v_4\}$ and edges $f=\{v_2,v_1\},g=\{v_2,v_3\},h=\{v_2,v_4\}$, where $T=\{v_1,v_2,v_3,v_4\}$. Note that $N|\{f,g,h,t\}$ is the graft matroid of (G'',T). Since $\Theta(p_1)$ is a circuit of M_Ω, it is also a circuit of M', and as it contains $\{f,p_1,t\}$ and all of its other edges belong to $C-\{f,g,h,t\}$, it follows that $\{f,p_1,t\}$ is a cycle of N. Similarly, $\{g,p_2,t\}$ and $\{h,p_3,t\}$ are also cycles of N. As a consequence, $N|\{f,g,h,t,p_1,p_2,p_3\}$ is the graft matroid of the plain quadrum (G',T) obtained from (G'',T) after adding $p_1=\{v_3,v_4\}$, $p_2=\{v_4,v_1\}$ and $p_3=\{v_1,v_3\}$.

Notice that N has no loop, because $\big(C-\{f,g,h,t\}\big)\cup\{e\}$ contains no circuit of M', for each $e\in E(N)$. Therefore, since $\{f,f'\},\{g,g'\},\{h,h'\}$ are odd circuits in (M',Σ'), they are also odd circuits in (N,Γ). Moreover, as $\Theta(p_1)\triangle\Theta(p_1')$ is an odd cycle of (M_Ω,Σ_Ω), it is also an odd cycle of (M',Σ'), and because it contains $\{p_1,p_1'\}$ and all of its other edges belong to $C-\{f,g,h,t\}$, it follows that $\{p_1,p_1'\}$ is an odd circuit of (N,Γ). Thus, N is the graft matroid of the graft (G,T) obtained from (G',T) after adding edges f',g',h',p_1' parallel to f, g, h, p_1, respectively. (See below for an illustration.) Moreover,

$$|\Gamma \cap \{f, f'\}| = |\Gamma \cap \{g, g'\}| = |\Gamma \cap \{h, h'\}| = |\Gamma \cap \{p_1, p_1'\}| = 1,$$

and we do not know whether or not p_2, p_3, t belong to Γ. This means that (G, T, Γ) is a super quadrum.



From Proposition 21 we get that (N, Γ), and therefore $(M, E(M))$, has an $(F_7, E(F_7))$ minor, as desired. $\square$

5.3. **Splitting triangles.** Take a signed matroid (N, Γ). For $R \subseteq E(N)$, we write $(N, \Gamma)|R := (N, \Gamma) \setminus (E(N) - R)$. We say that $\{e_1, \ldots, e_6\}$ is in *induced odd* K_4 of (N, Γ) if $N|\{e_1, \ldots, e_6\}$ is an induced K_4 in which every triangle is odd in $(N, \Gamma)|\{e_1, \ldots, e_6\}$. A consequence of Remark 25 is the following:

Corollary 28. *Suppose (h1)-(h7) hold. Then $(M, E(M))/\Omega$ does not have an induced odd K_4.*

Proof. Suppose, for a contradiction, that $\{e_1, \ldots, e_6\}$ is an induced odd K_4 of $(M, E(M))/\Omega$, whose odd triangles are $\{e_1, e_2, e_3\}, \{e_1, e_4, e_5\}, \{e_2, e_4, e_6\}, \{e_3, e_5, e_6\}$. It then follows from Remark 25 that these are also triangles of M, implying in turn that $\{e_1, \ldots, e_6\}$ is an induced K_4 of M, thereby contradicting (h6). $\square$

Remark 29. *Suppose (h1)-(h7) hold. If S is a triangle of M_Ω, then $|S \cap \{f, g, h\}| \leq 1$.*

Proof. Let S be a triangle of M_Ω. It follows from Proposition 23 (b) that $|S \cap \{f, g, h\}| \leq 2$. Suppose, for a contradiction, that $|S \cap \{f, g, h\}| = 2$. We may assume that $S \cap \{f, g, h\} = \{f, g\}$. By Remark 22 (a), one of $S, S \cup \{\Omega\}$ is a circuit of M, so one of $S, S \triangle \{f, f'\}$ is a triangle of M. But then $S \cup \{\Omega, f, f', g, g'\}$ is an induced K_4 of M, a contradiction to (h6). $\square$

We are now ready to prove the main result of this section:

Proposition 30. *Suppose (h1)-(h7) hold, and $(M, E(M))$ does not have $(F_7, E(F_7))$ as a minor. Then for every element $e \in C - \{f, g, h\}$, there exist triangles S_f, S_g, S_h of M_Ω such that $S_f \cap S_g = S_g \cap S_h = S_h \cap S_f = \{e\}$, and for each $z \in \{f, g, h\}$,*

- *S_z is z-splitting, and*
- *if $S_z \cap \{f, g, h\} = \emptyset$, then S_z is odd in $(M_\Omega, \Sigma_\Omega)$.*

Proof. Take an element $e \in C - \{f, g, h\}$. Denote by T_1, T_2, T_3 the three triangles of M containing e, whose existence is guaranteed by (h2). Recall that $T_1 \cap T_2 = T_2 \cap T_3 = T_3 \cap T_1 = \{e\}$. Since $e \notin \{\Omega, f, f', g, g', h, h'\}$, $\Omega \notin T_1 \cup T_2 \cup T_3$. Therefore, since M is a simple matroid and M/Ω is a loopless matroid whose non-trivial parallel classes are precisely $\{f,f'\},\{g,g'\},\{h,h'\}$, it follows that T_1,T_2,T_3 are also triangles of M/Ω; note that they are odd triangles of $(M,E(M))/\Omega$. For each $i\in[3]$, let S_i be the triangle corresponding to T_i in the simplification M_Ω. We will show that, after a relabeling, S_1,S_2,S_3 are the desired three triangles.

1. $S_1\cap S_2=S_2\cap S_3=S_3\cap S_1=\{e\}$. *Moreover, for each* $i\in[3]$, S_i *is a splitting triangle, and if* $S_i\cap\{f,g,h\}=\emptyset$*, then* S_i *is odd in* (M_Ω,Σ_Ω)*.*

Subproof. Suppose, for a contradiction, that $\{e\}\subsetneq S_1\cap S_2$. Since $\{e\}=T_1\cap T_2$, we may assume that $f\in S_1\cap S_2$, $f\in T_1$ and $f'\in T_2$. However, since $\{f,f',\Omega\}$ is a triangle of M, it follows that $T_1\cup T_2\cup\{\Omega\}$ is an induced K_4 of M, a contradiction to (h6). Thus, $S_1\cap S_2=\{e\}$ and similarly, $S_2\cap S_3=S_3\cap S_1=\{e\}$. Take an index $i\in[3]$. Clearly, if $S_i\cap\{f,g,h\}=\emptyset$, then $S_i=T_i$ and therefore S_i is an odd triangle of (M_Ω,Σ_Ω). Moreover, since $e\in S_i$, we get from Corollary 24 that S_i is a splitting triangle. $\Diamond$

It therefore suffices to show that no two of S_1,S_2,S_3 split the same element of $\{f,g,h\}$. Suppose, for a contradiction, that S_1,S_2 are f-splitting. Since these triangles are f-splitting, it follows that $S_1\cap\{g,h\}=S_2\cap\{g,h\}=\emptyset$, and by (1), $S_1\cap S_2=\{e\}$.

Fix an index $i\in[2]$. Let us carefully label the elements of S_i-e. If $f\in S_i$, then let $p_i:=f$ and q_i the element in $S_i-\{e,f\}$. Otherwise, $f\notin S_i$. Because S_i is f-splitting, $S_i\cap C=\{e\}$ and $S_i\triangle C$ is the union of two disjoint circuits of M_Ω. That is, the elements of $S_i-\{e\}$ are f-splitting, and for a labeling p_i,q_i of these elements, $S_i\triangle C$ is the disjoint union of $\Theta(p_i)$ and $C\triangle\Theta(q_i)$.

Since M_Ω is a simple matroid, when $f\notin\{p_1,p_2\}$, we get that $\Theta(p_1)-p_1\neq\Theta(p_2)-p_2$. We may therefore assume that $f\neq p_2$ and, if $f\neq p_1$, $(\Theta(p_2)-p_2)-(\Theta(p_1)-p_1)\neq\emptyset$.

2. $M_\Omega|\big(C\cup\{p_1,q_1,p_2,q_2\}\big)$ *is the cycle matroid of a simple graph G described as follows: for some integers* n,k *such that* $n-2\geq k\geq 3$*,*

- $V(G)=\{v_1,\dots,v_n\}$,
- $C=\big\{\{v_1,v_2\},\{v_2,v_3\},\dots,\{v_{n-1},v_n\},\{v_n,v_1\}\big\}$,
- $e=\{v_1,v_2\}$, $p_1=\{v_2,v_k\}$, $q_1=\{v_1,v_k\}$, $p_2=\{v_2,v_{k+1}\}$, $q_2=\{v_1,v_{k+1}\}$*, and*
- $f\in\big\{\{v_2,v_3\},\dots,\{v_{k-1},v_k\}\big\}$ *and* $g,h\in\big\{\{v_{k+1},v_{k+2}\},\dots,\{v_{n-1},v_n\},\{v_n,v_1\}\big\}$.

Subproof. Let $n:=|C|$. Clearly, $M_\Omega|C$ is the cycle matroid of the simple graph G_1 on vertices $\{v_1,\dots,v_n\}$ whose edges are $\{v_1,v_2,\},\{v_2,v_3\},\dots,\{v_{n-1},v_n\},\{v_n,v_1\}$. We assume that $e=\{v_1,v_2\}$ and the edges of $\Theta(q_1)-q_1$ appear consecutively on the graph circuit. Then there is an integer $k\geq 3$ such that $\Theta(q_1)-q_1=\big\{\{v_1,v_2\},\{v_2,v_3\},\dots,\{v_{k-1},v_k\}\big\}$. Note that $f\in\big\{\{v_2,v_3\},\dots,\{v_{k-1},v_k\}\big\}$, and $f=p_1$ if and only if $k=3$.

Let G_2 be the graph obtained from G_1 after adding the edge $q_1=\{v_1,v_k\}$, and if $k>3$, the edge $p_1=\{v_2,v_k\}$. Note that $M_\Omega|\big(C\cup\{p_1,q_1\}\big)$ is the cycle matroid of the simple graph G_2. Consider the set $\Theta(p_2)-p_2$. As $f\in\Theta(p_2)$, we have $(\Theta(p_2)-p_2)\cap\big\{\{v_2,v_3\},\dots,\{v_{k-1},v_k\}\big\}\neq\emptyset$, and as $(\Theta(p_2)-p_2)-(\Theta(p_1)-p_1)\neq\emptyset$ when $k>3$, we have $(\Theta(p_2)-p_2)\cap\big\{\{v_k,v_{k+1}\},\dots,\{v_{n-1},v_n\},\{v_n,v_1\}\big\}\neq\emptyset$. After possibly rearranging the edges of G_2 within series classes $\big\{\{v_2,v_3\},\dots,\{v_{k-1},v_k\}\big\}$ and $\big\{\{v_k,v_{k+1}\},\dots,\{v_{n-1},v_n\},\{v_n,v_1\}\big\}$,

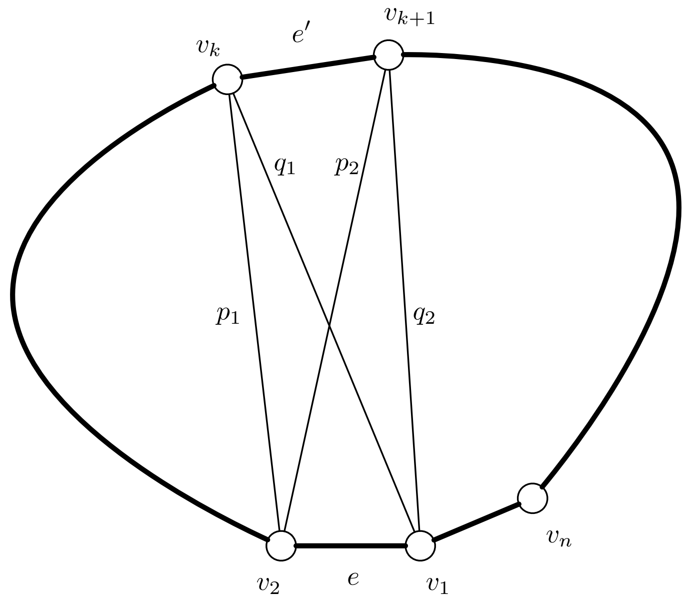


Figure 4. An illustration of graph G, where the edges in C are bold.

we may assume that the edges of $\Theta(p_2)-p_2$ appear consecutively on the circuit C. So there are indices $i,j\in[n]$ such that

$$\Theta(p_2)-p_2=\big\{\{v_i,v_{i+1}\},\dots,\{v_{j-1},v_j\}\big\}$$

where $k-1\geq i\geq 2$ and $n-1\geq j\geq k+1$.

Let G_3 be the graph obtained from G_2 after adding the edge $p_2=\{v_i,v_j\}$. Note that $M_\Omega|\big(C\cup\{p_1,q_1,p_2\}\big)$ is the cycle matroid of the simple graph G_3. We will show that $i=2$ and $j=k+1$. Consider the following circuit of G_3:

$$\{p_2\}\cup\big\{\{v_i,v_{i+1}\},\dots,\{v_{k-1},v_k\}\big\}\cup\{q_1\}\cup\big\{\{v_1,v_n\},\{v_n,v_{n-1}\},\dots,\{v_{j+1},v_j\}\big\}.$$

This circuit contains edges f,g,h and has $n-(i-1)-(j-k)+2$ many edges. It therefore follows from the minimality of C in (h5) that $n-(i-1)-(j-k)+2\geq n$, implying in turn that $i=2$ and $j=k+1$. Now let G be the graph obtained from G_3 after adding the edge $q_2=\{v_1,v_{k+1}\}$. It is clear that $M_\Omega|\big(C\cup\{p_1,q_1,p_2,q_2\}\big)$ is the cycle matroid of the simple graph G, which is the desired graph. $\Diamond$

By Remark 29, edges g,h do not lie in a triangle of G, so in fact $n-3\geq k$. Let $e':=\{v_k,v_{k+1}\}\in E(G)=E(M_\Omega)$ and note that $\{e,p_1,q_1,p_2,q_2,e'\}$ is an induced K_4 of M_Ω. Let

$$(N,\Gamma):=(M_\Omega,\Sigma_\Omega)|\{e,p_1,q_1,p_2,q_2,e'\}.$$

The triangle $S_2=\{e,p_2,q_2\}$, being disjoint from $\{f,g,h\}$, is odd in (N,Γ), and if $f\neq p_1$, then the triangle $S_1=\{e,p_1,q_1\}$ would also be odd in (N,Γ). Since M has no induced K_4 by (h6), it follows from Corollary 28 that exactly two of $\{e,p_1,q_1\},\{e',p_1,p_2\},\{e',q_1,q_2\}$ are even in (N,Γ). Thus, if $f\neq p_1$, then $\{e\}$ is a signature for (N,Γ), and if $f=p_1$, then one of $\{e\},\{e,f\},\{e,q_1\}$ is a signature for (N,Γ).

3. $f \neq p_1$.

Subproof. Suppose, for a contradiction, that $f = p_1$. We will show that $\{p_2, q_1\}$ is a Type I configuration. Recall that p_2, q_1 are f-splitting, and since $n-3 \geq k$, it follows that $C - \big(\Theta(p_2) \cup \Theta(q_1) \cup \{f, g, h\}\big) \neq \emptyset$. Moreover, $|\Theta(p_2)| = |\Theta(q_1)| = 3$. If $\{e, q_1\}$ is a signature for (N, Γ), then $\{e, f, q_1, p_2, q_2, e'\} \triangle \{f, f'\}$ is an induced odd K_4 of $(M, E(M))/\Omega$, which cannot occur by Corollary 28. Thus, one of $\{e\}, \{e, f\}$ is a signature for (N, Γ). Either way, we see that $\Theta(p_2) \triangle \Theta(q_1)$ is odd cycle of $(M_\Omega, \Sigma_\Omega)$. Thus, $\{p_2, q_1\}$ is a Type I configuration. But then Proposition 26 implies that $(M, E(M))$ has an $(F_7, E(F_7))$ minor, a contradiction to our hypothesis. $\Diamond$

Recall that p_1, q_1, p_2, q_2 are f-splitting, and since $n-3 \geq k$,

$$C - \big(\Theta(p_2) \cup \Theta(q_1) \cup \{f, g, h\}\big) = C - \big(\Theta(p_1) \cup \Theta(q_2) \cup \{f, g, h\}\big) \neq \emptyset.$$

We also know that $\{e\}$ is a signature for (N, Γ). It can be readily seen that if $|\Theta(p_1) - p_1|$ is odd, then $\{p_2, q_1\}$ is a Type I configuration, and otherwise, $\{p_1, q_2\}$ is a Type I configuration. Either way, we get from Proposition 26 that $(M, E(M))$ has an $(F_7, E(F_7))$ minor, thereby contradicting our hypothesis. Therefore, S_1 and S_2 cannot both be f-splitting. Similarly, no two of S_1, S_2, S_3 split the same element. Among these triangles, let S_f be the f-splitting one, S_g the g-splitting one, and S_h the h-splitting one. These are the desired triangles, and the proof of Proposition 30 is finished. $\square$

5.4. **Proof of Proposition 12.** We may assume that (h1)-(h7) hold. To remind the reader why, assume that (h1)-(h2), as well as the setup conditions (h3)-(h5), hold. If M has an induced K_4, then we are done. Otherwise, (h6) holds. We will prove that $(M, E(M))$ has $(F_7, E(F_7))$ as a minor, thereby finishing the proof of Proposition 12. Suppose otherwise. It then follows from Proposition 23 (c) that (h7) holds.

1. $|C| \geq 6$.

Subproof. Suppose otherwise. By (h7), $|C|$ is even. Thus, for some $t \in E(M_\Omega)$, $C = \{f, g, h, t\}$. By Proposition 30, there is an h-splitting triangle $\{t, h, p\}$, where p is h-splitting. But then $\{p, f, g\}$ is a triangle of M_Ω, thereby contradicting Remark 29. $\Diamond$

2. *There is an f-splitting triangle S where $|S \cap C| = 1$ and S is odd in $(M_\Omega, \Sigma_\Omega)$.*

Subproof. Suppose otherwise. By (1), there are distinct elements $e_1, e_2, e_3 \in C - \{f, g, h\}$. Fix an index $i \in [3]$. Then by Proposition 30 and our contrary assumption, e_i is contained in an f-splitting triangle S_i such that $S_i \cap C = \{e_i, f\}$. By Remark 22 (a), one of $S_i, S_i \cup \{\Omega\}$ is a circuit of M, implying in turn that one of $S_i, S_i \triangle \{f, f'\}$ is a triangle of M. By (h2), f and f' are each in exactly 3 triangles of M, a common one being $\{\Omega, f, f'\}$. Hence, it cannot be that each one of S_1, S_2, S_3 is a triangle of M or that each one of $S_1 \triangle \{f, f'\}, S_2 \triangle \{f, f'\}, S_3 \triangle \{f, f'\}$ is a triangle of M. We may therefore assume that $S_1, S_2 \triangle \{f, f'\}$ are triangles of M. In other words, $S_1, S_2 \cup \{\Omega\}$ are circuits of M, so by Remark 25, S_1 is an odd triangle and S_2 is an even triangle of $(M_\Omega, \Sigma_\Omega)$. Let p_1 be the element in $S_1 - \{e_1, f\}$ and p_2 the element in $S_2 - \{e_2, f\}$. Then p_1, p_2 are f-splitting elements for which $\Theta(p_1) = S_1$ and $\Theta(p_2) = S_2$. Since $e_3 \in C - (S_1 \cup S_2 \cup \{f, g, h\})$, it follows that $\{p_1, p_2\}$ is a Type I configuration. By Proposition 26, $(M, E(M))$ has an $(F_7, E(F_7))$ minor, a contradiction. $\Diamond$

Write $S = \{e, p_1, p_1'\}$ where $C \cap S = \{e\}$ and $e \in \Theta(p_1')$. Note that $\Theta(p_1') \triangle \Theta(p_1) = S$, so it is an odd cycle of $(M_\Omega, \Sigma_\Omega)$. As M_Ω is simple, there is an element $t \in \Theta(p_1) - \{f, p_1\}$. Note that $t \in \Theta(p_1')$. By Proposition 30, t is contained in a g-splitting triangle S_2 and an h-splitting triangle S_3. Pick the g-splitting element $p_2 \in S_2$ for which $t \in \Theta(p_2)$ and the h-splitting element $p_3 \in S_3$ for which $t \in \Theta(p_3)$. Then $\{p_1, p_1', p_2, p_3\}$ is a Type II configuration. By Proposition 27, $(M, E(M))$ has an $(F_7, E(F_7))$ minor, which is a contradiction. This finishes the proof of Proposition 12. $\square$

6. Even cycle matroids

Let G be a graph and $\Gamma \subseteq E(G)$. The signed matroid $(\text{cycle}(G), \Gamma)$ is identified as (G, Γ) and is simply referred to as a *signed graph*. Zaslavsky [22] proved that the even cycles of (G, Γ) are the cycles of a (binary) matroid that we call the *even cycle matroid* of (G, Γ) and denote by $\text{ecycle}(G, \Gamma)$. Notice that every signature of (G, Γ) is a cocycle of $\text{ecycle}(G, \Gamma)$.

Given a graph H and a new edge label e, denote by $H + e$ any graph obtained from H after adding e as a loop. The following folklore result states that matroids one contraction away from being graphic are even cycle matroids:

Remark 31. *Take a binary matroid M and an element $e \in E(M)$ such that $M/e = \text{cycle}(H)$, for some graph H. If Γ is a cocycle of M containing e, then $M = \text{ecycle}(H + e, \Gamma)$.*

Proof. Let Γ be a cocycle of M containing e. Let $C \subseteq E(M)$. We need to show that C is a cycle of M if, and only if, C is an even cycle of $(H + e, \Gamma)$. ($\Rightarrow$) Suppose first that C is a cycle of M. Since $|C \cap \Gamma|$ is even, it suffices to show that C is a cycle of $H + e$. If $e \in C$, then $C - e$ is a cycle of M/e by Remark 22 (c), so it is also a cycle of H, implying in turn that C is a cycle of $H + e$. Otherwise, $e \notin C$, so C is a cycle of M/e by Remark 22 (d), implying in turn that C is a cycle of H, and therefore, of $H + e$. ($\Leftarrow$) Suppose conversely that C is an even cycle of $(H + e, \Gamma)$. Assume first that $e \notin C$. Then C is a cycle of H, so it is also a cycle of M/e, implying by Remark 22 (b) that either C or $C \cup \{e\}$ is a cycle of M. Since $|C \cap \Gamma|$ is even, $e \in \Gamma$ and $e \notin C$, it follows that $|(C \cup \{e\}) \cap \Gamma|$ is odd, and because Γ is a cocycle of M, $C \cup \{e\}$ cannot be a cycle of M. Thus, C is a cycle of M. Assume in the remaining case that $e \in C$. Then $C - e$ is a cycle of H, so it is also a cycle of M/e, and thus by Remark 22 (b), one of $C - e, C$ is a cycle of M. Since $e \in C \cap \Gamma$ and $|C \cap \Gamma|$ is even, it follows that $|(C - e) \cap \Gamma|$ is odd. Therefore, because Γ is a cocycle of M, $C - e$ cannot be a cycle of M, and as a result, C is a cycle of M. $\square$

6.1. Even cycle matroids and connectivity.

Let G be a graph. For a subset $X \subseteq E(G)$, we denote by $V_G(X)$ the ends of the edges in X, and by $G[X]$ the subgraph on vertices $V_G(X)$ and edges X.

Let (G, Γ) be a signed graph, where G is connected. If (G, Γ) has no odd circuit, then $\text{ecycle}(G, \Gamma) = \text{cycle}(G)$ and therefore, any spanning tree of G is a basis for $\text{ecycle}(G, \Gamma)$. Otherwise, when (G, Γ) has an odd circuit, $T \cup \{e\}$ is a basis for ecycle(G,Γ), where T is a spanning tree of G, and $e \in E(G) - T$ is chosen so that $T \cup \{e\}$ contains an odd circuit of (G,Γ).

The next remark describes the connectivity function for even cycle matroids.

Remark 32 ([7]). *Let (G,Γ) be a signed graph, where G is connected. Take a non-empty and proper subset $X \subseteq E(G)$ where both $G[X]$ and $G[\bar{X}]$ are connected. Then*

$$\lambda_{ecycle(G,\Gamma)}(X) \leq \lambda_{cycle(G)}(X) + 1 = \left|V_G(X) \cap V_G(\bar{X})\right|.$$

Proof. The equation is a routine exercise (see [12] Lemma 8.1.7 for a proof). To prove the inequality, let $E := E(G)$, $M :=$ cycle(G) and $M' :=$ ecycle(G,Γ). Denote by r, r' the rank functions of M, M', respectively. If (G,Γ) has no odd circuit, then $M = M'$, so $r = r'$, implying in turn that $\lambda_M = \lambda_{M'}$. We may therefore assume that (G,Γ) has an odd circuit. What we argued above implies that $r'(E) = r(E)+1$, $r'(X) \in \{r(X), r(X)+1\}$ and $r'(\bar{X}) \in \{r(\bar{X}), r(\bar{X})+1\}$, so

$$\lambda_{M'}(X) = r'(X) + r'(\bar{X}) - r'(E) \leq r(X) + 1 + r(\bar{X}) + 1 - r(E) - 1 = \lambda_M(X) + 1,$$

as required. □

A connected graph on at least 3 vertices is 2-*connected* if it remains connected after deleting any vertex. A 2-connected graph on at least 4 vertices is 3-*connected* if it remains connected after deleting any pair of vertices. For a graph G, denote the set of all loops by loops(G). Given a signed graph (G,Γ), denote by si(G,Γ) the signed graph obtained after deleting all even loops, deleting all odd loops except for one, and deleting all but one edge from each class of parallel edges in G of the same parity in (G,Γ).

Proposition 33. *Let (G,Γ) be a signed graph that has an odd loop e, and let $N :=$ ecycle(G,Γ). Then,*

- *a. if N is simple and cosimple, e is the unique loop of G, parallel edges of G have distinct parities in (G,Γ), and G does not have edges in series,*
- *b. if N is internally 4-connected and $|E(N)| \geq 8$, then $G \setminus e$ is 3-connected,*
- *c. si(N) = ecycle(si(G,Γ)),*
- *d. if si(N) is internally 4-connected and $|E(\text{si}(N))| \geq 8$, then $G \setminus$ loops(G) is 3-connected.*

Proof. **(a)** Suppose N is simple and cosimple. Then N has no cycle of size at most 2 and no two elements in series. In particular, (G,Γ) does not have an even loop or an even cycle of size two, and G does not have two edges in series. Since e is an odd loop, there cannot be another odd loop. As a result, e is the unique loop of G and parallel edges of G have distinct parities in (G,Γ). **(b)** Suppose N is internally 4-connected. In particular, N is simple and cosimple. Thus, (a) implies that in G, no two edges are in series, e is the unique loop, and every parallel class has size at most two. Since $|E(G \setminus e)| \geq 7$, we get that $G \setminus e$ has at least 4 vertices. It suffices to show that when $G \setminus e$ is connected, then it is 3-connected. We first show that $G \setminus e$ is 2-connected. Suppose, for a contradiction, that there is a non-trivial partition X, Y of $E(G \setminus e)$ such that $|V_{G\setminus e}(X) \cap V_{G\setminus e}(Y)| = 1$. Since $|X| + |Y| \geq 7$, after possibly interchanging the roles of X and Y, we may assume that $|X| \geq 2$. Let $\bar{X} := Y \cup \{e\}$. Then $|\bar{X}| \geq 2$. Assuming the end of e belongs to $V_{G\setminus e}(Y)$, we see that $G, G[X], G[\bar{X}]$ are connected. Hence, Remark 32 implies that $\lambda_N(X) \leq |V_G(X) \cap V_G(\bar{X})| = 1$, so $(X, \bar{X})$ is a 2-separation of N, a contradiction as N is 3-connected. It remains to show that $G \setminus e$ is 3-connected. Suppose, for a contradiction, that there is a non-trivial partition X, Y of $E(G \setminus e)$ such that $|V_{G\setminus e}(X) \cap V_{G\setminus e}(Y)| = 2$, $|V_{G\setminus e}(X)| \geq 3$ and $|V_{G\setminus e}(Y)| \geq 3$. Since $G \setminus e$ is 2-connected, it follows that $(G \setminus e)[X], (G \setminus e)[Y]$ are connected, implying in turn that $|X| \geq 2$ and $|Y| \geq 2$. In fact, since $G \setminus e$ does not have two edges in series, we have $|X| \geq 3$ and $|Y| \geq 3$. Because $|X| + |Y| \geq 7$, we may assume that $|X| \geq 4$. Let $\bar{X} := Y \cup \{e\}$. Then $|\bar{X}| \geq 4$. Assuming the end of e belongs to $V_{G\setminus e}(Y)$, we see that $G, G[X], G[\bar{X}]$ are connected. Thus, by Remark 32, we get that $\lambda_N(X) \leq |V_G(X) \cap V_G(\bar{X})| = 2$, so $(X, \bar{X})$ is a 3-separation of N, a contradiction as N is internally 4-connected. **(c)** is immediate. **(d)** Let $(G', \Gamma') := \mathrm{si}(G, \Gamma)$. We may assume that e is also an odd loop of (G', Γ'). By (c), $\mathrm{si}(N) = \mathrm{ecycle}(G', \Gamma')$, so from (b) we get that $G' \setminus e$ is 3-connected. As G is obtained from G' by adding loops and edges parallel to existing ones, we get that $G \setminus \mathrm{loops}(G)$ is also 3-connected. □

6.2. **Even cycle matroids that are graphic.** Here we characterize, under relevant conditions, when an even cycle matroid is graphic. A complete and technical answer to this problem was obtained by Shih in his PhD thesis [20] but was never published in a refereed journal – our arguments will not rely on this characterization. We will need the following seminal result of Whitney [21]:

Theorem 34. *Let G, G' be graphs over the same edge set such that cycle(G) = cycle(G'). If $G \setminus$ loops(G) is 3-connected, then $G \setminus$ loops$(G) = G' \setminus$ loops(G') and loops(G) = loops(G').*

Let (G, Γ) be a signed graph, and take a vertex $v \in V(G)$. We say v is a *blocking vertex* if every non-loop odd circuit of (G, Γ) uses v. It follows from Proposition 2 that v is a blocking vertex if, and only if, there is a signature contained in $\delta_G(v) \cup \mathrm{loops}(G)$.

Remark 35. *Let (G, Γ) be a signed graph. If (G, Γ) has a blocking vertex, then ecycle(G, Γ) is graphic.*

Proof. Let v be a blocking vertex. After possibly resigning, we may assume that $\Gamma \subseteq \delta_v(G) \cup \mathrm{loops}(G)$. We may also assume that every odd loop is incident to v and every even loop is incident to another vertex. Let H be the graph obtained from G after splitting v into vertices v_1, v_2 such that every edge in $\delta_G(v) \cap \Gamma$ is incident with v_1, every edge in $\delta_G(v) - \Gamma$ is incident with v_2, and every odd loop has ends v_1, v_2. It can be readily checked that $\mathrm{ecycle}(G, \Gamma) = \mathrm{cycle}(H)$. □

Provided an odd loop and 3-connectedness, we can guarantee the converse also holds:

Proposition 36. *Let (G, Γ) be a signed graph that has an odd loop and $G \setminus$ loops(G) is 3-connected. If ecycle(G, Γ) is graphic, then (G, Γ) has a blocking vertex.*

Proof. Set $E := E(G)$ and let $e \in E$ be an odd loop of (G, Γ). Let H be a graph with edge set E such that $\mathrm{ecycle}(G, \Gamma) = \mathrm{cycle}(H)$. As e is not an even loop of (G, Γ), e is not a loop of H; let v_1, v_2 be the ends of e in H. Since the even circuits of (G, Γ) are precisely the circuits of H we have, for $C \subseteq E$, the following correspondence:

- C is an odd circuit of (G,Γ) if, and only if, C is a v_1v_2-path of H,
- C is an even circuit of (G,Γ) if, and only if, C is a circuit of H.

Let G' be the graph obtained from H after identifying vertices v_1 and v_2; call the identified vertex v. Let $\Gamma' := \delta_H(v_1)$. Then the correspondence above implies that $\text{cycle}(G') = \text{cycle}(G)$ and $\text{ecycle}(G',\Gamma') = \text{ecycle}(G,\Gamma)$. Since $G \setminus \text{loops}(G)$ is 3-connected, it follows from Theorem 34 that $G' \setminus \text{loops}(G') = G \setminus \text{loops}(G)$ and $\text{loops}(G) = \text{loops}(G')$. After changing the ends of the loops of G', if necessary, we may assume that $G' = G$. Since $\text{ecycle}(G,\Gamma') = \text{ecycle}(G,\Gamma)$, Γ' is a signature of (G,Γ) and as $\Gamma' \subseteq \delta_G(v) \cup \text{loops}(G)$, we see that v is a blocking vertex of (G,Γ). ☐

6.3. **Blocking pairs.** Let (G,Γ) be a signed graph. Take disjoint $I, J \subseteq E(G)$. If I contains an odd circuit, we define $(G,\Gamma)/I \setminus J := (G/I \setminus J, \emptyset)$. Otherwise, by Proposition 2, there is a signature Γ' that is disjoint from I, and we define $(G,\Gamma)/I \setminus J := (G/I \setminus J, \Gamma' - J)$. We call $(G,\Gamma)/I \setminus J$ a *minor* of (G,Γ). Notice that minors are defined only up to resigning, and since $\text{cycle}(G)/I \setminus J = \text{cycle}(G/I \setminus J)$, the signed graph $(G,\Gamma)/I \setminus J$ represents $(\text{cycle}(G), \Gamma)/I \setminus J$. We also have the following relationship:

Remark 37 ([13], page 21)**.** *Take a signed graph (G,Γ) and disjoint $I, J \subseteq E(G)$. Then ecycle$(G,\Gamma)/I \setminus J =$ ecycle$((G,\Gamma)/I \setminus J)$.*

Take vertices u, v of G. We say u and v form a *blocking pair* if every non-loop odd circuit of (G,Γ) uses either u or v. We see from Proposition 2 that u and v form a blocking pair if, and only if, there is a signature contained in $\delta_G(u) \cup \delta_G(v) \cup \text{loops}(G)$.

Proposition 38. *Let (G,Γ) be a signed graph with an odd loop and without a blocking vertex, and let $N :=$ ecycle(G,Γ). If e is a non-loop edge of G such that*

- *$|E(si(N/e))| \geq 8$,*
- *$si(N/e)$ is internally 4-connected,*
- *$si(N/e)$ is graphic,*

then the ends of e form a blocking pair.

Proof. Let $(G',\Gamma') := (G,\Gamma)/e$. By Remark 37, $N/e = \text{ecycle}(G',\Gamma')$. Notice that (G',Γ') also has an odd loop. Therefore, as $\text{si}(N/e)$ is internally 4-connected and $|E(\text{si}(N/e))| \geq 8$, it follows from Proposition 33 (d) that $G' \setminus \text{loops}(G')$ is 3-connected. Since $\text{si}(N/e)$ is graphic, so is N/e, and so $\text{ecycle}(G',\Gamma')$ is graphic. Putting these together, we get from Proposition 36 that (G',Γ') has a blocking vertex w, that is, every non-loop odd circuit of (G',Γ') uses w. Since (G,Γ) does not have a blocking vertex, w is the vertex in $G' = G/e$ obtained after identifying the ends of e in G. Thus, every non-loop odd circuit of (G,Γ) uses one of the ends of e, implying that the ends of e form a blocking pair of (G,Γ), as required. ☐

7. Proof of Proposition 13

Suppose (h1)-(h2) hold and there are distinct elements e_1, e_2, e_3, e_4 of M such that, for each $i \in [4]$, $\text{si}(M/e_i)$ is internally 4-connected and is the cycle matroid of a graph where the three edges $\Lambda(e_i)$ are incident to the same vertex. Assuming $|E(M)| \geq 12$, we need to show M is one deletion away from being graphic. Recall that, for each $i \in [4]$, M/e_i is a loopless matroid with exactly three non-trivial parallel classes, and these classes have cardinality two, so $|E(\mathrm{si}(M/e_i))| = |E(M)| - 4 \geq 12 - 4$:

1. *For each $i \in [4]$, $|E(si(M/e_i))| \geq 8$.*

By (h1), M is simple, so we may assume that $\{e_1, e_2, e_3\}$ is not a triangle of M. Denote the three triangles of M containing e_1 by $\{e_1, f, f'\}, \{e_1, g, g'\}, \{e_1, h, h'\}$ where $\Lambda(e_1) = \{f, g, h\}$; the existence of these triangles is guaranteed by (h2). Recall that the non-trivial parallel classes of M/e_1 are $\{f, f'\}, \{g, g'\}, \{h, h'\}$. As it is the case for $\mathrm{si}(M/e_1)$, we know that M/e_1 also is the cycle matroid of a graph H where f, g, h are incident to the same vertex, say $v \in V(H)$. Notice that H is a loopless graph with exactly three non-trivial parallel classes $\{f, f'\}, \{g, g'\}, \{h, h'\}$. In particular, v is the only vertex common to any two of f, g, h. Let Γ be a cocycle of M that contains e_1. By Remark 31, $M = \mathrm{ecycle}(H + e_1, \Gamma)$. Clearly e_1 is an odd loop of $(H + e_1, \Gamma)$, and therefore, $\{f, f'\}, \{g, g'\}, \{h, h'\}$ are odd circuits of this signed graph. As a result, if $(H + e_1, \Gamma)$ has a blocking vertex, then v must be the one, and if it has a blocking pair, then v must belong to the pair. If v is a blocking vertex, then M is graphic by Remark 35, and we are done. Otherwise,

2. *$(H + e_1, \Gamma)$ does not have a blocking vertex.*

Consider the two edges e_2, e_3 of H. Since $\{e_1, e_2, e_3\}$ is not a triangle of M, edges e_2, e_3 are not parallel. Since (1) and (2) hold, we may use Proposition 38 to conclude that, for $j \in \{2, 3\}$, the ends of e_j form a blocking pair of $(H + e_1, \Gamma)$. In particular, $e_2 \cap e_3 = \{v\}$. Write $e_2 = \{v, u\}$ and $e_3 = \{v, w\}$.

3. *$H \setminus \{u, v, w\}$ is connected.*

Subproof. Let $H' := H/e_2$ and $(H' + e_1, \Gamma') := (H + e_1, \Gamma)/e_2$. By Remark 37, $\mathrm{ecycle}(H' + e_1, \Gamma') = M/e_2$. Since (1) holds, we may use Proposition 33 (d) to conclude that $H' \setminus \mathrm{loops}(H')$ is 3-connected. In particular, if uv is the vertex of H' corresponding to the ends of e_2, the graph $H' \setminus \mathrm{loops}(H') \setminus \{uv, w\}$ is connected. As a result, $H \setminus \{u, v, w\}$ is connected. $\Diamond$

Since $\{v, u\}$ and $\{v, w\}$ are blocking pairs, every non-loop odd circuit of $(H + e_1, \Gamma)$ uses either v or both u, w. As the non-trivial parallel classes of H are incident with v, there is at most one edge with ends u, w.

4. *H has an edge Ω with ends u, w, and every non-loop odd circuit of $(H + e_1, \Gamma)$ uses either v or the edge Ω.*

Subproof. Let C be a non-loop odd circuit C of $(H + e_1, \Gamma)$ such that $v \notin V(C)$. Then $\{u, w\} \subseteq V(C)$. It suffices to show that C contains an edge whose ends are u and w. Suppose otherwise. Let x, y be the two neighbors of u in $H[C]$ – note $x, y \in V(H) - \{u, v, w\}$ by our contrary assumption. Thus by (3), there is an xy-path $P \subseteq E(H)$ that is disjoint from $\{u, v, w\}$. Consider the two cycles $C_1 := \{u, x\} \cup \{u, y\} \cup P$ and $C_2 := C \triangle C_1$. Since C_1 is disjoint from the blocking pair $\{v, w\}$, and C_2 is disjoint from the blocking pair $\{v, u\}$, it follows that both C_1, C_2 are even in $(H + e_1, \Gamma)$, implying in turn that $C = C_1 \triangle C_2$ is also even in $(H + e_1, \Gamma)$, contradicting our choice of C. $\Diamond$

Therefore, $(H+e_1,\Gamma)\backslash\Omega$ has v as a blocking vertex. By Remark 37, we have $M\backslash\Omega = \text{ecycle}((H+e_1,\Gamma)\setminus\Omega)$, so it follows from Remark 35 that $M\setminus\Omega$ is graphic, as required. □

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