

Resistant sets in the unit hypercube

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December 22, 2017

Abstract

Ideal matrices and clutters are prevalent in Combinatorial Optimization, ranging from balanced matrices, clutters of T -joins, to clutters of rooted arborescences. Most of the known examples of ideal clutters are combinatorial in nature. In this paper, rendered by the recently developed theory of cuboids, we provide a different class of ideal clutters, one that is geometric in nature. The advantage with this new class of ideal clutters is that it allows for infinitely many ideal minimally nonpacking clutters. We characterize the densest ideal minimally nonpacking clutters of the class.

1 Introduction

Let E be a finite set of *elements*, and let \mathcal{C} be a family of subsets of E called *members*. If no member is contained in another one, then \mathcal{C} is a *clutter* over *ground set* E [14]. The *incidence matrix* of \mathcal{C} , denoted $M(\mathcal{C})$, is the matrix whose columns are labeled by E and whose rows are the incidence vectors of the members. We say that \mathcal{C} is *ideal* if the set covering polyhedron $\{x \in \mathbb{R}^E : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}$ is integral [12]. Ideal clutters are prevalent in the literature:

- $M(\mathcal{C})$ is totally unimodular [21] or balanced [7] (also see [10], Chapter 6),
- \mathcal{C} is the clutter of T -joins of a graft [16] (also see [10], Theorems 1.21 and 2.1),
- \mathcal{C} is the clutter of odd circuits of a signed graph without an odd- K_5 minor [18],
- \mathcal{C} is the clutter of rooted arborescences of a directed graph [13, 17] (also see [4]).

All of these clutters are ideal because the members conform to a combinatorial pattern. In this paper, we come up with a different class of clutters that is ideal because a geometric pattern is followed.¹ To present this new class of ideal clutters, we need to move to a different, yet equivalent framework.

Take an integer $n \geq 1$. A *cuboid* is a clutter over ground set $[2n] := \{1, \dots, 2n\}$ where every member C satisfies the following:

$$|C \cap \{2i - 1, 2i\}| = 1 \quad \forall i \in [n].$$

¹This idea first appeared in a paper by Jon Lee [23].

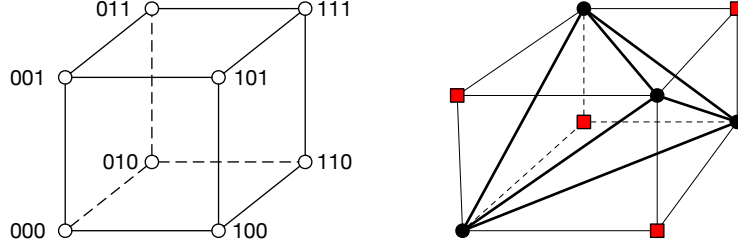


Figure 1: An illustration of the coordinate system, and the convex hull of $R_{1,1}$.

For instance, the clutter $Q_6 := \{\{2, 4, 6\}, \{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 5\}\}$ of triangles of K_4 is a cuboid.

When is a cuboid ideal?

This question is actually equivalent to asking when a clutter is ideal, even though cuboids form a special class of clutters [1]. To answer it, we need to view cuboids as vertex subsets of the unit n -dimensional hypercube. To this end, denote by $\{0, 1\}^n$ the vertices of the n -dimensional unit hypercube. Take a set $S \subseteq \{0, 1\}^n$. We will think of the points in S as *feasible points* and the points in \overline{S} as *infeasible points*. The *cuboid* of S , denoted $\text{cuboid}(S)$, is the clutter over ground set $[2n]$ whose members have incidence vectors

$$(x_1, 1 - x_1, \dots, x_n, 1 - x_n) \quad x \in S.$$

For instance, Q_6 is the cuboid of the set $R_{1,1} := \{000, 110, 101, 011\} \subseteq \{0, 1\}^3$.

Theorem 1.1 ([1], Theorem 1.6). *Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then $\text{cuboid}(S)$ is ideal if, and only if, S is cube-ideal.²*

We say that S is *cube-ideal* if the convex hull of the feasible points is described by inequalities of the form

$$\begin{aligned} 0 \leq x_i \leq 1 & \quad i \in [n] & \quad (\text{hypercube inequalities}) \\ \sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1 & \quad I, J \subseteq [n], I \cap J = \emptyset & \quad (\text{generalized set covering inequalities}). \end{aligned}$$

For instance, the set $R_{1,1}$ is cube-ideal (see Figure 1), so by Theorem 1.1, Q_6 is an ideal cuboid.

By Theorem 1.1, our question above is equivalent to the following:

When is a set cube-ideal?

To answer it, take a coordinate $i \in [n]$. To *twist S at coordinate i* is to replace S by the set

$$S \Delta e_i := \{x \Delta e_i : x \in S\}.$$

As twists correspond to the change of variables $x_i \mapsto 1 - x_i, i \in [n]$, and the hypercube and generalized set covering inequalities are closed under these transformations, if a set is cube-ideal then so is every twist of it.

²The dual of this statement was proved in [19, 26], where idealness of $0, \pm 1$ matrices was reduced to idealness of $0, 1$ matrices.

Given a point $x \in \{0, 1\}^n$, the *induced clutter of S with respect to x* is the clutter over ground set $[n]$ whose members are

$$\text{ind}(S\Delta x) := \text{the minimal sets of } \{C \subseteq [n] : \chi_C \in S\Delta x\}.$$

Notice that if x is feasible, then $\text{ind}(S\Delta x) = \{\emptyset\}$. We refer to the induced clutters $\text{ind}(S\Delta x)$, $x \in S$ as *trivial*, and to $\text{ind}(S\Delta x)$, $x \in \overline{S}$ as *nontrivial*. Notice that the nontrivial induced clutters pick up local information about the set S .

Theorem 1.2 ([1], Theorem 1.8). *A set is cube-ideal if, and only if, every nontrivial induced clutter is ideal.*

This is the key to generating a new class of ideal clutters.

1.1 Resistant sets

Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. We say that S is *resistant* if for every nontrivial induced clutter, the members are pairwise disjoint. For instance, the set $R_{1,1} = \{000, 110, 101, 011\}$ is resistant as its nontrivial induced clutters are equal to $\{\{1\}, \{2\}, \{3\}\}$. Clearly, if a set is resistant, then so is every twist of it.

Remark 1.3. *A clutter whose members are pairwise disjoint is ideal.*

An immediate consequence of Theorem 1.2 and Remark 1.3 is that,

Corollary 1.4. *Resistant sets are cube-ideal.*

Combining this with Theorem 1.1, we obtain a new class of ideal clutters:

Corollary 1.5. *Cuboids of resistant sets are ideal clutters.*

Resistant sets form a rich class of cube-ideal sets. We will see several basic classes of resistant sets in §2; let us display one of them here. Denote by G_n the *skeleton graph* of $\{0, 1\}^n$ whose vertices are the points in $\{0, 1\}^n$ and two points u, v are adjacent if they differ in exactly one coordinate. A *feasible component* of S is a (connected) component of the vertex induced subgraph $G_n[S]$, while an *infeasible component* of S is a component of $G_n[\overline{S}]$. We will prove the following in §2:

Theorem 1.6. *Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$, where every infeasible component is a hypercube or has maximum degree at most two. Then S is resistant.*

There are several binary operations that preserve resistance, one way or another. Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0, 1\}^{n_1}, S_2 \subseteq \{0, 1\}^{n_2}$. Define the *product*, *coproduct* and *reflective product* of S_1, S_2 as

$$\begin{aligned} S_1 \times S_2 &:= \{(x, y) \in \{0, 1\}^{n_1} \times \{0, 1\}^{n_2} : x \in S_1 \text{ and } y \in S_2\} \\ S_1 \oplus S_2 &:= \{(x, y) \in \{0, 1\}^{n_1} \times \{0, 1\}^{n_2} : x \in S_1 \text{ or } y \in S_2\} \\ S_1 * S_2 &:= (S_1 \times S_2) \cup (\overline{S_1} \times \overline{S_2}), \end{aligned}$$

respectively. Notice that $\overline{S_1 \oplus S_2} = \overline{S_1} \times \overline{S_2}$ and $\overline{S_1 * S_2} = \overline{S_1} * \overline{S_2}$. The following theorem is proved in §2:

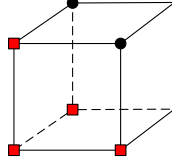


Figure 2: An illustration of a fragile set. Round points are feasible and square points are infeasible.

Theorem 1.7. Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0, 1\}^{n_1}, S_2 \subseteq \{0, 1\}^{n_2}$. Then the following statements hold:

- (1) If S_1 is resistant, then so is $S_1 \times \{0, 1\}^{n_2}$.
- (2) If S_1, S_2 are resistant, then so is $S_1 \oplus S_2$.
- (3) If $S_1, \overline{S_1}, S_2, \overline{S_2}$ are resistant, then so are $S_1 * S_2, \overline{S_1} * \overline{S_2}$.

Take a coordinate $i \in [n]$. The set obtained from $S \cap \{x : x_i = 0\}$ after dropping coordinate i is called the 0-restriction of S over coordinate i , and the set obtained from $S \cap \{x : x_i = 1\}$ after dropping coordinate i is called the 1-restriction of S over coordinate i . A restriction of S is a set obtained after a series of 0- and 1-restrictions. The projection of S over coordinate i is the set obtained from S after dropping coordinate i . A projection of S is a set obtained after a series of single projections. A minor of S is what is obtained after a series of restrictions and projections. A minor is *proper* if at least one minor operation is applied.

We will see in §3 that if a set is resistant, then so is every minor of it. So what are the excluded minors defining resistance? We say that two sets S, S' are *isomorphic*, and denote it by $S \cong S'$, if one is obtained from the other after twisting and relabeling some coordinates. Take a set $F \subseteq \{0, 1\}^3$ such that

$$F \cap \{000, 100, 010, 001, 101, 011\} = \{101, 011\}.$$

We refer to F , and any set isomorphic to it, as *fragile* (see Figure 2). Observe that F is not resistant because its induced clutter with respect to the origin has intersecting members $\{1, 3\}, \{2, 3\}$. We will prove the following characterization of resistant sets in §3:

Theorem 1.8. Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then the following statements are equivalent:

- (i) S is resistant,
- (ii) S has no fragile restriction and no $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}, k \geq 4$ isomorphic restriction,³
- (iii) S has no fragile minor.

We will prove this in §3 ($\mathbf{0}^k, \mathbf{1}^k$ denote the k -dimensional vectors whose entries are 0, 1, respectively). In that section, we will also justify our choice of the terms “resistant” and “fragile”, and prove the following statement:

Theorem 1.9. Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then in time $O(n^2|S|^2)$, one can test whether or not S is resistant.

³Hereinafter, the adjective “isomorphic” will be omitted from “isomorphic restriction” and “isomorphic minor”.

1.2 When do cuboids of resistant sets have the packing property?

Let \mathcal{C} be a clutter over ground set E . We say that \mathcal{C} *packs* if the maximum number of pairwise disjoint members is equal to the minimum number of elements needed to intersect every member. For instance, the clutter of edges of a bipartite graph over the vertex set packs [22], while the clutter Q_6 does not [25, 27]. Given disjoint subsets $I, J \subseteq E$, the *minor* of \mathcal{C} obtained after *deleting* I and *contracting* J is the clutter over ground set $E - (I \cup J)$ whose members are

$$\mathcal{C} \setminus I/J := \text{the minimal sets of } \{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}.$$

A minor is *proper* if $I \cup J \neq \emptyset$. We say that \mathcal{C} has the *packing property* if every minor of it, including \mathcal{C} itself, packs [11]. A consequence of Lehman's theorem [24] is that clutters with the packing property are ideal [11]. The converse however is not true, even for cuboids of resistant sets, as Q_6 shows. So when do cuboids of resistant sets have the packing property?

Take an integer $n \geq 1$. Two points $a, b \in \{0, 1\}^n$ are *antipodal* if $a + b = \mathbf{1}$. We say that $S \subseteq \{0, 1\}^n$ is *polar* if either there are antipodal feasible points or the feasible points all agree on a coordinate. We say that S is *strictly polar* if every restriction of it, including S itself, is polar. We need the following result:

Theorem 1.10 ([1], Theorem 1.11). *Let S be a set. Then $\text{cuboid}(S)$ has the packing property if, and only if, S is strictly polar and every induced clutter of it has the packing property.*

The reader can easily verify the following remark:

Remark 1.11. *A clutter whose members are pairwise disjoint has the packing property.*

As a consequence,

Corollary 1.12. *Take an integer $n \geq 1$ and a resistant set $S \subseteq \{0, 1\}^n$. Then $\text{cuboid}(S)$ has the packing property if, and only if, S is strictly polar.*

So, when is a resistant set strictly polar? We will prove in §4 the following statement:

Theorem 1.13. *Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then the following statements are equivalent: (i) S is resistant and strictly polar, (ii) in every restriction of S , either there are antipodal feasible points or the feasible points form a hypercube.*

We say that S is *strictly nonpolar* if it is nonpolar, but every proper restriction is polar. Notice that a set is strictly polar if, and only if, it has no strictly nonpolar restriction. We will prove in §4 that,

Theorem 1.14. *Take an integer $n \geq 1$ and a resistant set $S \subseteq \{0, 1\}^n$. Then S is strictly nonpolar if, and only if, $\text{cuboid}(S)$ is an ideal minimally nonpacking clutter.*

A clutter is *minimally nonpacking* if it does not pack, but every proper minor does. This theorem motivates us even further to pose the following question:

Question 1.15. *What are the strictly nonpolar sets that are resistant?*

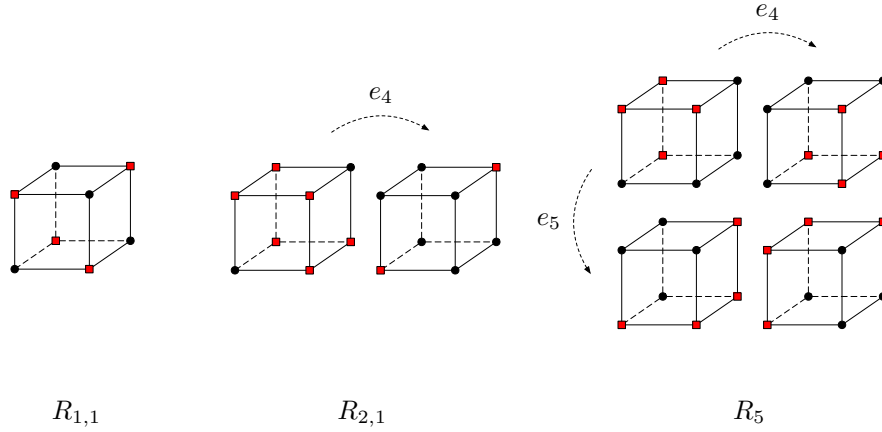


Figure 3: An illustration of $R_{1,1}$, $R_{2,1}$, R_5 .

Even though we are not able to answer this question, we can characterize the resistant strictly nonpolar sets of maximum possible cardinality. To elaborate, take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Observe that if S is nonpolar, then $|S| \leq 2^{n-1}$, and if equality holds, we say that S is *half-dense*. In [1], strictly nonpolar sets are extensively studied, and among the many identified/generated examples, a significant portion are half-dense (out of the 745 strictly nonpolar sets of dimension at most 7 enumerated, 74 are half-dense). Out of their examples, an infinite class and a sporadic instance played a significant role, namely the half-dense strictly nonpolar sets

$$R_{k,1} := \{\mathbf{0}^{k+1}, \mathbf{1}^{k+1}\} * \{0\} \quad (k \geq 1) \quad \text{and} \quad R_5 := R * \{0\},$$

where

$$R = \left\{ \sum_{i=1}^d e_i, \mathbf{1}^4 - \sum_{i=1}^d e_i : d \in [4] \right\}.$$

We will see in §2 that $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are also resistant sets (see Figure 3). As a first step towards answering Question 1.15, we prove the following, which is the main result of this paper:

Theorem 1.16. $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are, up to isomorphism, the only half-dense strictly nonpolar sets that are resistant.

As an application, we prove the following:

Theorem 1.17. Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0, 1\}^{n_1}, S_2 \subseteq \{0, 1\}^{n_2}$, where $S_1, \overline{S_1}, S_2, \overline{S_2}$ are nonempty and resistant. Then $S_1 * S_2$ is strictly polar if, and only if, $S_1 * S_2$ has none of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction.

The proofs of these theorems can be found in §6, and tools needed to prove them are provided in §4 and §5.

2 Basic resistant classes and resistance-preserving operations

In this section, we exhibit three basic classes of resistant sets as well as three operations that preserve resistance, prove Theorems 1.6 and 1.7, and show as a consequence that $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are resistant sets.

Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. A *minimal point* of S is simply a point in S of minimal support. We will need the following observation:

Proposition 2.1. *Take an integer $n \geq 1$, a set $S \subseteq \{0, 1\}^n$, and an infeasible component $K \subseteq \bar{S}$. Then for each $x \in K$, $\text{ind}(S \Delta x) = \text{ind}(\bar{K} \Delta x)$.*

Proof. After a twisting, if necessary, we may assume that $x = \mathbf{0}$. Then we need to show that S and \bar{K} have the same set of minimal points. Notice that $S \subseteq \bar{K}$.

Claim 1. *If y is a minimal point of \bar{K} , then $y \in S$. In particular, the minimal points of \bar{K} are also minimal points of S .*

Proof of Claim. Since y is a minimal point of \bar{K} , there is a hypercube $H \subseteq \{0, 1\}^n$ such that $\mathbf{0} \in H$ and $H \cap \bar{K} = \{y\}$. In particular, $H - \{y\} \subseteq K$, implying in turn that y is adjacent to a point of K . Since K is an infeasible component of S and $y \notin K$, it follows that $y \in S$. \diamond

Claim 2. *The minimal points of S are also minimal points of \bar{K} .*

Proof of Claim. Let y be a minimal point of S . Then there is a hypercube $H \subseteq \{0, 1\}^n$ such that $\mathbf{0} \in H$ and $H \cap S = \{y\}$. Thus, as $G_n[H - \{y\}]$ is connected and contains $\mathbf{0}$, it follows that $H - \{y\} \subseteq K$. As a result, $H \cap \bar{K} = \{y\}$, implying in turn that y is a minimal point of \bar{K} , as required. \diamond

Claims 1 and 2 finish the proof. \square

As a consequence, the infeasible components of a set dictate whether or not a set is resistant:

Corollary 2.2. *Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then the following statements are equivalent:*

- (i) S is resistant,
- (ii) for every infeasible component K , the set \bar{K} is resistant.⁴

We will need the following proposition as well:

Proposition 2.3. *Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then the following statements hold:*

- (1) *If the infeasible points form a hypercube, then S is resistant.*
- (2) *If every infeasible point has at most two infeasible neighbors, then S is resistant.*

⁴Notice that Proposition 2.1, together with Theorem 1.2, gives us an analogue of Remark 2.2 for cube-idealness: “ S is cube-ideal if, and only if, \bar{K} is cube-ideal for every infeasible component K .” This fact was noticed earlier in a much more general setting by Angulo, Ahmed, Dey, Kaibel [6], and their insight, believe it or not, had a large impact on this paper and our other papers [1, 3, 5].

Proof. **(1)** As the infeasible points form a hypercube, S does not have a 2-dimensional restriction with exactly one feasible point. Thus, a hypercube of dimension at least 2 cannot have exactly one feasible point. In particular, for each infeasible point x , a minimal point of $S \triangle x$ cannot have support of cardinality at least 2. This implies that for each infeasible point x , the members of $\text{ind}(S \triangle x)$ have cardinality one, implying in particular that the members of $\text{ind}(S \triangle x)$ are pairwise disjoint. Therefore, S is resistant. **(2)** Take an infeasible point x . We need to show that the members of $\text{ind}(S \triangle x)$ are pairwise disjoint. After a possible twisting, we may assume that $x = \mathbf{0}$, and as x has at most two infeasible neighbors, we may assume after a possible relabeling that $e_3, e_4, \dots, e_n \in S$. As a result, $\text{ind}(S \triangle x) = \text{ind}(S)$ has $\{3\}, \{4\}, \dots, \{n\}$ as members. Hence, as its ground set is $[n]$, $\text{ind}(S)$ cannot have intersecting members, as required. \square

As a consequence,

Proof of Theorem 1.6. Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$, where every infeasible component is a hypercube or has maximum degree at most two. It then follows from Corollary 2.2 and Proposition 2.3 that S is resistant, as required. \square

There are also resistant sets that are the union of an arbitrary number of hypercubes that are pairwise arbitrarily far apart:

Proposition 2.4. *The following statements hold:*

(1) *Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$ that is the disjoint union of two nonempty hypercubes and contains antipodal feasible points. Then S is resistant.*

(2) *Take an integer $p \geq 3$ and another integer $n \geq \binom{p}{2}$. Let $(P_{ij} : i, j \in [p], j > i)$ be a partition of $[n]$ into $\binom{p}{2}$ nonempty parts. For each $i \in [p]$, let H_i be the hypercube of points $y \in \{0, 1\}^n$ such that*

$$y_k = 0 \quad \forall k \in \bigcup (P_{ij} : j > i) \quad \text{and} \quad y_k = 1 \quad \forall k \in \bigcup (P_{ji} : j < i).$$

Then $H_1 \cup H_2 \cup \dots \cup H_p$ is resistant.

Proof. **(1)** Suppose that S is the disjoint union of two nonempty hypercubes H_1 and H_2 , and S contains antipodal points a, b . Since H_1, H_2 are nonempty and disjoint, neither of them can contain both a and b , so we may assume that $a \in H_1$ and $b \in H_2$. Take an infeasible point x , if any. We need to show that the members of $\text{ind}(S \triangle x)$ are pairwise disjoint. Since hypercubes (resp. antipodal points) remain hypercubes (resp. antipodal) after twists, we may assume that $x = \mathbf{0}$. It therefore suffices to show that the minimal points of S have disjoint supports. As H_1 (resp. H_2) is a nonempty hypercube, it has a unique point of minimal support, say a' (resp. b'). Notice that $\{a', b'\}$ are the minimal points of S . As $a' \leq a$ and $b' \leq b$, and as a, b are antipodal, it follows that a', b' have disjoint supports. Thus, S is resistant.

(2) Take an infeasible point x . For each $i \in [p]$, let x^i be the point in the hypercube $H_i \triangle x$ of minimal support. It suffices to show that x^1, \dots, x^p have pairwise disjoint supports. To this end, pick a coordinate $k \in [n]$. Choose $i, j \in [p]$ such that $j > i$ and $k \in P_{ij}$. Then $x_k^i + x_k^j = 1$ and $x_k^\ell = 0$ for all $\ell \in [p] - \{i, j\}$.

As a result, exactly one of x^1, \dots, x^p has coordinate k in its support, and since this is true for every $k \in [n]$, it follows that x^1, \dots, x^p have pairwise disjoint supports, as required. \square

Let us see one last example of resistant sets; this class arises from clutters. We say that S is *up-monotone* if for all $x, y \in \{0, 1\}^n$ such that $x \geq y$, if y is feasible then so is x . The up-monotone set *associated with* a clutter \mathcal{C} over ground set E is

$$\{\chi_C : C \subseteq E \text{ contains a member}\} \subseteq \{0, 1\}^E.$$

An element of a clutter is *free* if it is not contained in any member. We will need the following observation:

Remark 2.5 ([1], Remark 4.6). *Take an integer $n \geq 1$, an up-monotone set $S \subseteq \{0, 1\}^n$, and a point $x \in \{0, 1\}^n$. Then $\text{ind}(S \triangle x)$ is, after deleting free elements, equal to $\text{ind}(S) / \{i \in [n] : x_i = 1\}$.*

As a result,

Proposition 2.6. *Let \mathcal{C} be a clutter whose members are pairwise disjoint, and let S be the associated up-monotone set. Then S is resistant.*

Proof. Take an infeasible point x . Then after deleting free elements, $\text{ind}(S \triangle x)$ is a contraction minor of $\text{ind}(S) = \mathcal{C}$, by Remark 2.5. As the members of \mathcal{C} are pairwise disjoint, the members of $\text{ind}(S \triangle x)$ are also pairwise disjoint. So S is resistant. \square

Let us now introduce three binary operations that can be used to generate resistant sets starting from smaller ones. Let E_1, E_2 be disjoint finite sets, and let $\mathcal{C}_1, \mathcal{C}_2$ be clutters over ground sets E_1, E_2 , respectively. The *product* and *coproduct* of $\mathcal{C}_1, \mathcal{C}_2$ are the clutters over ground set $E_1 \cup E_2$ whose members are

$$\begin{aligned} \mathcal{C}_1 \times \mathcal{C}_2 &:= \{C_1 \cup C_2 : C_1 \in \mathcal{C}_1, C_2 \in \mathcal{C}_2\} \\ \mathcal{C}_1 \oplus \mathcal{C}_2 &:= \text{the minimal sets of } \{C : C \in \mathcal{C}_1\} \cup \{C : C \in \mathcal{C}_2\}, \end{aligned}$$

respectively. We need the following result:

Proposition 2.7 ([1], Remark 5.4 and Proposition 5.6). *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0, 1\}^{n_1}, S_2 \subseteq \{0, 1\}^{n_2}$. Then, viewing $\text{ind}(S_1)$ and $\text{ind}(S_2)$ as clutters over disjoint ground sets, we have that*

$$\begin{aligned} \text{ind}(S_1 \times S_2) &= \text{ind}(S_1) \times \text{ind}(S_2) \\ \text{ind}(S_1 \oplus S_2) &= \text{ind}(S_1) \oplus \text{ind}(S_2) \\ \text{ind}(S_1 * S_2) &= \begin{cases} \{\emptyset\} & \text{if } \mathbf{0} \in S_1 \text{ and } \mathbf{0} \in S_2 \\ \{\emptyset\} & \text{if } \mathbf{0} \in \overline{S_1} \text{ and } \mathbf{0} \in \overline{S_2} \\ \text{ind}(S_1) \oplus \text{ind}(\overline{S_2}) & \text{if } \mathbf{0} \in \overline{S_1} \text{ and } \mathbf{0} \in S_2 \\ \text{ind}(\overline{S_1}) \oplus \text{ind}(S_2) & \text{if } \mathbf{0} \in S_1 \text{ and } \mathbf{0} \in \overline{S_2}. \end{cases} \end{aligned}$$

Now let us prove Theorem 1.7, telling us how to generate resistant sets starting from smaller ones:

Proof of Theorem 1.7. Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0, 1\}^{n_1}, S_2 \subseteq \{0, 1\}^{n_2}$. **(1)** Assume that S_1 is resistant. We need to show that $S_1 \times \{0, 1\}^{n_2}$ is resistant. To this end, take a point $(x, y) \in \{0, 1\}^{n_1} \times \{0, 1\}^{n_2}$. It suffices to show that the members of $\text{ind}((S_1 \times \{0, 1\}^{n_2}) \Delta(x, y))$ are pairwise disjoint. Well, by Proposition 2.7,

$$\begin{aligned} \text{ind}((S_1 \times \{0, 1\}^{n_2}) \Delta(x, y)) &= \text{ind}((S_1 \Delta x) \times \{0, 1\}^{n_2}) = \text{ind}(S_1 \Delta x) \times \text{ind}(\{0, 1\}^{n_2}) \\ &= \text{ind}(S_1 \Delta x) \times \{\emptyset\} \\ &= \text{ind}(S_1 \Delta x), \end{aligned}$$

and as S_1 is resistant, the result follows. **(2)** Assume that S_1, S_2 are resistant. We need to show that $S_1 \oplus S_2$ is resistant. To this end, take a point $(x, y) \in \{0, 1\}^{n_1} \times \{0, 1\}^{n_2}$. By Proposition 2.7,

$$\text{ind}((S_1 \oplus S_2) \Delta(x, y)) = \text{ind}((S_1 \Delta x) \oplus (S_2 \Delta y)) = \text{ind}(S_1 \Delta x) \oplus \text{ind}(S_2 \Delta y).$$

As S_1, S_2 are resistant, the members of $\text{ind}(S_1 \Delta x), \text{ind}(S_2 \Delta y)$ are pairwise disjoint, implying in turn that the members of $\text{ind}((S_1 \oplus S_2) \Delta(x, y)) = \text{ind}(S_1 \Delta x) \oplus \text{ind}(S_2 \Delta y)$ are pairwise disjoint, so $S_1 \oplus S_2$ is resistant.

(3) Assume that $S_1, \overline{S_1}, S_2, \overline{S_2}$ are resistant. We need to show that $S_1 * S_2, \overline{S_1} * \overline{S_2} = \overline{S_1} * S_2$ are resistant. Clearly, it suffices to show that $S_1 * S_2$ is resistant. Pick an arbitrary point $(x, y) \in \{0, 1\}^{n_1} \times \{0, 1\}^{n_2}$. Then $(S_1 * S_2) \Delta(x, y) = (S_1 \Delta x) * (S_2 \Delta y)$, so by Proposition 2.7, $\text{ind}((S_1 * S_2) \Delta(x, y))$ is one of

$$\{\emptyset\} \quad \text{or} \quad \text{ind}(S_1 \Delta x) \oplus \text{ind}(\overline{S_2} \Delta y) \quad \text{or} \quad \text{ind}(\overline{S_1} \Delta x) \oplus \text{ind}(S_2 \Delta y)$$

(notice that $\overline{S_2 \Delta y} = \overline{S_2} \Delta y$ and $\overline{\overline{S_1} \Delta x} = \overline{S_1} \Delta x$). As $S_1, \overline{S_1}, S_2, \overline{S_2}$ are all resistant, it follows that $\text{ind}(S_1 \Delta x), \text{ind}(\overline{S_2} \Delta y), \text{ind}(\overline{S_1} \Delta x), \text{ind}(S_2 \Delta y)$ each have only pairwise disjoint members. As a result, the members of $\text{ind}((S_1 * S_2) \Delta(x, y))$ are pairwise disjoint. Thus, $S_1 * S_2$ is resistant. \square

We are now ready to prove the following:

Remark 2.8. $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are resistant sets.

Proof. Take an integer $k \geq 1$. Recall that $R_{k,1} = \{\mathbf{0}^{k+1}, \mathbf{1}^{k+1}\} * \{0\}$. By Proposition 2.4 (1), $\{\mathbf{0}^{k+1}, \mathbf{1}^{k+1}\}$ is resistant, and by Theorem 1.6, $\overline{\{\mathbf{0}^{k+1}, \mathbf{1}^{k+1}\}}$ is resistant. It is clear that $\{0\}, \{1\}$ are resistant too. As a result, by Theorem 1.7 (3), $R_{k,1}$ is resistant. To prove that R_5 is resistant, recall that $R_5 = R * \{0\}$ for

$$R = \left\{ \sum_{i=1}^d e_i, \mathbf{1}^4 - \sum_{i=1}^d e_i : d \in [4] \right\}.$$

Notice that R is a circuit of length 8, and $R \cong \overline{R}$. As a result, Theorem 1.6 implies that both R, \overline{R} are resistant. Hence, Theorem 1.7 (3) implies that R_5 is resistant. \square

3 Testing resistance in polynomial time

Here we prove Theorems 1.8 and 1.9. We need the following easy remark:

Remark 3.1 ([1], Remark 2.11(i)). *Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then an induced clutter of a minor of S is a minor of an induced clutter of S .*

If the members of a clutter are pairwise disjoint, then so are the members of any minor of it. This fact, combined with the preceding remark, implies the following:

Remark 3.2. *If a set is resistant, then so is any minor of it.*

We are now ready to prove Theorem 1.8, providing two characterizations of resistant sets:

Proof of Theorem 1.8. Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. We need to prove that the following statements are equivalent:

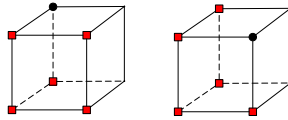
- (i) S is resistant,
- (ii) S has no fragile restriction and no $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}, k \geq 4$ restriction,
- (iii) S has no fragile minor.

(i) \Rightarrow (ii): As we noted already, fragile sets are not resistant. For each integer $k \geq 4$, the set $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}$ is not resistant either, because its induced clutter with respect to $e_1 + e_2$ has intersecting members $\{1, 2\}$ and $\{1, 3, 4, \dots, k\}$. Remark 3.2 now tells us that (i) implies (ii).

(ii) \Rightarrow (iii): Assume that S has a fragile minor. It suffices to show that S has either a fragile restriction or a $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}$ restriction, for some $k \in \{4, \dots, n\}$. We will need the following two claims:

Claim 1. *Suppose $R \subseteq \{0, 1\}^4$ has no fragile restriction and its projection over coordinate 4 is fragile. Then R is a twist of $\{\mathbf{0}^4, \mathbf{1}^4 - e_1\}$.*

Proof of Claim. For $i \in \{0, 1\}$, let $R_i \subseteq \{0, 1\}^3$ be the i -restriction of R over coordinate 4. Since the projection of R over 4 is fragile, it follows that $\{000, 100, 010, 001\} \subseteq \overline{R_0}$ and $\{000, 100, 010, 001\} \subseteq \overline{R_1}$. Moreover, as R_0 and R_1 are not fragile, we may assume that $R_0 \cap \{101, 011\} = \{011\}$ and $R_1 \cap \{101, 011\} = \{101\}$:



Once again, as R_0 and R_1 are not fragile, it follows that $110 \notin R_0 \cup R_1$. Since the 1-restriction of R over coordinate 1 is not fragile, we get that $111 \notin R_0$, and since the 1-restriction of R over coordinate 2 is not fragile, $111 \notin R_1$. Thus, R is a twist of $\{\mathbf{0}^4, \mathbf{1}^4 - e_1\}$, as claimed. \diamond

Claim 2. *Take an integer $k \geq 4$ and a set $R \subseteq \{0, 1\}^{k+1}$ without a $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}$ restriction. If the projection of R over coordinate $k + 1$ is $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}$, then R is a twist of $\{\mathbf{0}^{k+1}, \mathbf{1}^{k+1} - e_1\}$.*

Proof of Claim. For $i \in \{0, 1\}$, let $R_i \subseteq \{0, 1\}^k$ be the i -restriction of R over coordinate $k + 1$. Clearly, $R_i \subseteq \{\mathbf{0}^k, \mathbf{1}^k - e_1\}$ for each $i \in \{0, 1\}$. As equality cannot hold, we may assume that $R_0 \cap \{\mathbf{0}^k, \mathbf{1}^k - e_1\} = \{\mathbf{0}^k\}$ and $R_1 \cap \{\mathbf{0}^k, \mathbf{1}^k - e_1\} = \{\mathbf{1}^k - e_1\}$, implying in turn that R is a twist of $\{\mathbf{0}^{k+1}, \mathbf{1}^{k+1} - e_1\}$. \diamond

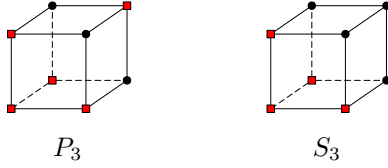


Figure 4: An illustration of P_3 and S_3 , the smallest non-cube-ideal sets.

Suppose a fragile minor of S is obtained after applying k single projections and $n - k - 3$ single restrictions, for some $k \in \{0, \dots, n - 3\}$. If $k = 0$, then S has a fragile restriction, so we are done. We may therefore assume that $k \geq 1$ and S has no fragile restriction. It follows from Claim 1 that S has a $\{\mathbf{0}^4, \mathbf{1}^4 - e_1\}$ minor obtained after applying $k - 1$ single projections and $n - k - 3$ single restrictions. If $k = 1$, then S has a $\{\mathbf{0}^4, \mathbf{1}^4 - e_1\}$ restriction. We may therefore assume that $k \geq 2$ and S has no $\{\mathbf{0}^4, \mathbf{1}^4 - e_1\}$ restriction. Now by repeatedly applying Claim 2, we see that S has one of $\{\mathbf{0}^\ell, \mathbf{1}^\ell - e_1\}$, $\ell \in \{5, \dots, k + 3\}$ as a restriction, as required.

(iii) \Rightarrow (i): Assume that S is not resistant. It suffices to prove that S has a fragile minor. After possibly twisting S , we may assume that $\mathcal{C} := \text{ind}(S)$ has intersecting members.

Claim 3. *There exist disjoint $I, J \subseteq [n]$ such that $\mathcal{C} \setminus I/J$ has ground set $\{x, y, z\}$ and has $\{x, z\}, \{y, z\}$ among its members.*

Proof of Claim. Since the minimal points of S do not have pairwise disjoint supports, \mathcal{C} has intersecting members. Among all pairs of intersecting members, pick an intersecting pair C_1, C_2 whose union is minimal. Our minimal choice of C_1, C_2 implies that every member of \mathcal{C} contained in $C_1 \cup C_2$ is either C_1 or C_2 or it contains $C_1 \triangle C_2$. Take elements $x \in C_1 - C_2$, $y \in C_2 - C_1$ and $z \in C_1 \cap C_2$. Let $I := [n] - (C_1 \cup C_2)$ and $J := [n] - (I \cup \{x, y, z\})$. It is easy to see that $\mathcal{C} \setminus I/J$ has ground set $\{x, y, z\}$ and has $\{x, z\}, \{y, z\}$ among its members. \diamond

Consider now the minor S' of S obtained after 0-restricting coordinates I and projecting away coordinates J . Since $\text{ind}(S) = \mathcal{C}$, it follows that $\text{ind}(S') = \mathcal{C} \setminus I/J$ has $\{x, z\}, \{y, z\}$ as members, implying in turn that S' is fragile. Thus, S has a fragile minor, as required. \square

Define $P_3 := \{110, 011, 101\} \subseteq \{0, 1\}^3$ and $S_3 := \{110, 011, 101, 111\} \subseteq \{0, 1\}^3$ (see Figure 4). Notice that the induced clutters of P_3 and S_3 with respect to the origin are equal to $\Delta_3 := \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$. As $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is an extreme point of its set covering polyhedron, Δ_3 is nonideal, so by Theorem 1.2, P_3, S_3 are not cube-ideal. In fact, it can be readily checked that P_3 and S_3 are the only non-cube-ideal sets of dimension at most 3. The following immediate remark now justifies our choice of the term ‘‘fragile’’:

Remark 3.3. *Take a set $S \subseteq \{0, 1\}^3$. Then S is fragile if, and only if, $S \cup X$ is isomorphic to either P_3, S_3 for some set $X \subseteq \{0, 1\}^3$ of cardinality at most one.*

Theorem 1.8 and Remark 3.3 together imply the following result, which justifies our choice of the term ‘‘resistant’’:

Theorem 3.4. *Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then S is resistant if, and only if, $S \cup X$ has no P_3, S_3 minor for all sets $X \subseteq \{0, 1\}^n$ of cardinality at most one.*

Proof. (\Leftarrow) Assume that S is not resistant. Then by Theorem 1.8 (iii), S has a fragile minor $S' \subseteq \{0, 1\}^3$. After a possible twisting and relabeling, we may assume that S is obtained after 0-restricting $I \subseteq [n] - \{1, 2, 3\}$ and projecting away $J \subseteq [n] - \{1, 2, 3\}$. Since S' is fragile, Remark 3.3 implies that there is a set $X' \subseteq \{0, 1\}^3$ of cardinality at most one such that $S' \cup X'$ is isomorphic to one of P_3, S_3 . Define $X \subseteq \{0, 1\}^n$ as follows: if $X' = \emptyset$ set $X := \emptyset$, otherwise set $X := \{x\}$ where $x \in \{0, 1\}^n$ is obtained from the point in X' by setting the coordinates in $I \cup J$ to 0. Then $S \cup X$ has an $S' \cup X'$ minor obtained after 0-restricting I and projecting away J . Since $S' \cup X'$ is isomorphic to one of P_3, S_3 , we get that $S \cup X$ has one of P_3, S_3 as a minor, as required. (\Leftarrow) Assume that $S \cup X$ has one of P_3, S_3 as minor, for some set $X \subseteq \{0, 1\}^n$ of cardinality at most one. Then there is a set $Y \subseteq \{0, 1\}^3$ of cardinality at most one such that S has one of $P_3 - Y, S_3 - Y$ as a minor. By Remark 3.3, both $P_3 - Y, S_3 - Y$ are fragile, so S has a fragile minor. Thus, by Theorem 1.8 (iii), S is not resistant. \square

Take an integer $n \geq 1$ and points $a, b \in \{0, 1\}^n$. Denote by $\text{dist}(a, b)$ the (*Hamming*) distance between a and b , that is, $\text{dist}(a, b)$ is the number of coordinates a, b disagree on. We will next prove Theorem 1.9, stating that the resistance of $S \subseteq \{0, 1\}^n$ can be tested in time $O(n^2|S|^2)$.

Proof of Theorem 1.9. We will take advantage of Theorem 1.8 (ii), stating that $S \subseteq \{0, 1\}^n$ is resistant if, and only if, it has no fragile restriction and no $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}, k \in \{4, \dots, n\}$ restriction. For $k \in \{3, 4, \dots, n\}$, consider the following algorithm:

1. for every pair of points x, y of S at distance $k - 1$,
 - (a) let $I := \{i \in [n] : x_i = y_i\}$,
 - (b) for every coordinate $i \in I$,
 - i. let $S' \subseteq \{0, 1\}^k$ be the restriction of S over coordinates $I - \{i\}$ containing (images of) x and y ,
 - ii. if $k = 3$ and S' is fragile, then output “ S has a fragile restriction”,
 - iii. if $k \geq 4$ and S' is isomorphic to $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}$, then output “ S has a $\{\mathbf{0}, \mathbf{1} - e_1\} \subseteq \{0, 1\}^k$ restriction”,
 - (c) if $k = 3$ and (ii) fails for every $i \in I$, then change the pair x, y ,
 - (d) if $k \geq 4$ and (iii) fails for every $i \in I$, then change the pair x, y ,
2. if $k = 3$ and (ii) fails for every pair x, y , then output “ S has no fragile restriction”,
3. if $k \geq 4$ and (iii) fails for every pair x, y , then output “ S has no $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}$ restriction”.

The correctness of this algorithm is clear; its running time is $\binom{|S|}{2} \times (n - k + 1)$. Thus, by Theorem 1.8 (ii), one can test whether or not S is resistant in time $\sum_{k=3}^n \binom{|S|}{2} \times (n - k + 1) = O(n^2|S|^2)$, as required. \square

4 Propagations

In this section, we prove three lemmas needed for all the forthcoming proofs, as well as prove Theorems 1.13 and 1.14. Before getting started, let us set up a few ingredients. Recall that for an integer $n \geq 1$, G_n is the skeleton graph of $\{0, 1\}^n$.

Remark 4.1. *For an integer $n \geq 1$, the following statements hold:*

- For points $a, b, c \in \{0, 1\}^n$, $\text{dist}(a, b) + \text{dist}(b, c) \geq \text{dist}(a, c)$.
- For points $a, b \in \{0, 1\}^n$, every (a, b) -path in G_n has at least $\text{dist}(a, b)$ many edges.

An (a, b) -path whose vertices are $a = v_0, v_1, \dots, v_k = b$ as traversed from a to b will be represented as the sequence (v_0, v_1, \dots, v_k) . The *length of a path* is the number of edges it has. An (a, b) -path of G_n is *straight* if it has length exactly $\text{dist}(a, b)$.

Remark 4.2. *Take an integer $n \geq 1$. Then the following statements hold:*

- Take distinct points a, b at Hamming distance $\ell \geq 1$, and let P be an (a, b) -path of G_n . Then P is straight if, and only if, there are ℓ distinct coordinates i_1, \dots, i_ℓ such that

$$P = (a, a \Delta e_{i_1}, a \Delta e_{i_1} \Delta e_{i_2}, \dots, a \Delta e_{i_1} \Delta e_{i_2} \Delta \dots \Delta e_{i_\ell} = b).$$

- Pick distinct points a, b, c such that $\text{dist}(a, b) + \text{dist}(b, c) = \text{dist}(a, c)$. If P is a straight (a, b) -path and Q a straight (b, c) -path of G_n , then $P \cup Q$ is a straight (a, c) -path of G_n .⁵

Take a set $S \subseteq \{0, 1\}^n$. A path in $G_n[S]$ is called a *feasible path*, and a path in $G_n[\overline{S}]$ is called an *infeasible path*. We say that S is *connected* if $G_n[S]$ is a connected graph. If every restriction of S , including S itself, is connected, then we say that S is *strictly connected*.

Proposition 4.3 ([1], Proposition 5.10). *Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then the following statements are equivalent:*

- S is strictly connected,
- S has no $\{\mathbf{0}^k, \mathbf{1}^k\}$, $k \geq 2$ restriction,
- for all distinct feasible points a and b , there is a straight feasible (a, b) -path.

⁵If P is an (a, b) -path and Q is a (b, c) -path, then $P \cup Q$ denotes the (a, c) -walk that first traverses the vertices of P from a to b , and then traverses the vertices of Q from b to c .

4.1 An example

Let us start with the following proposition which best illustrates the title of this section:

Proposition 4.4. *Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$, where for all $x \in \{0, 1\}^n$ and distinct $i, j \in [n]$, the following statement holds:*

$$\text{if } x, x\Delta e_i, x\Delta e_j \in S \text{ then } x\Delta e_i\Delta e_j \in S.$$

Then every component of S is a hypercube.

Proof. We claim that the following statement holds for all subsets $I \subseteq [n]$ of cardinality at least two:

$$(\star) \text{ If } x \in S \text{ and } x\Delta e_i \in S \text{ for each } i \in I, \text{ then } x\Delta \left(\sum_{i \in I} e_i\right) \in S.$$

We prove (\star) by induction on $|I| \geq 2$. The base case $|I| = 2$ is our hypothesis. For the induction step, assume that $k := |I| \geq 3$. After a possible twisting and relabeling, we may assume that $x = \mathbf{0}$ and $I = \{e_1, \dots, e_k\}$. To prove (\star) , we need to show that $\sum_{i=1}^k e_i \in S$. Let $y := \sum_{i=1}^{k-2} e_i$. If $k = 3$ then $y \in S$ by assumption, and if $k \geq 4$ then $y \in S$ by the induction hypothesis. Moreover, the induction hypothesis implies that $y\Delta e_{k-1}, y\Delta e_k \in S$. As a result, $\sum_{i=1}^k e_i = y\Delta e_{k-1}\Delta e_k \in S$ by our hypothesis, thereby completing the induction step. Thus, (\star) holds for every subset $I \subseteq [n]$ of cardinality at least two.

Take a component S' of S . Let d be the maximum number of feasible neighbors of a point in S' . If $d \leq 1$, then $|S'| \in \{1, 2\}$, so S' is clearly a hypercube. Otherwise, $d \geq 2$. After a possible twisting and relabeling, we may assume that $\mathbf{0}, e_1, \dots, e_d \in S'$. Then for all subsets $I \subseteq [d]$ of cardinality at least two, $\sum_{i \in I} e_i \in S$ by (\star) . As a result,

$$\{x \in \{0, 1\}^n : x_j = 0, j \in [n] - [d]\} \subseteq S'.$$

Since every feasible point in S' has at most d feasible neighbors, equality holds above, so S' is a hypercube, as required. \square

So the condition “if $x, x\Delta e_i, x\Delta e_j \in S$ then $x\Delta e_i\Delta e_j \in S$ ” has a *propagating effect*, ensuring that every feasible component is a hypercube. As a consequence,

Corollary 4.5. *Take an integer $n \geq 1$ and a set $S \subseteq \{0, 1\}^n$. Then every component of S is a hypercube if, and only if, S has no $\{00, 10, 01\}$ restriction.*

Proof. (\Rightarrow) If every feasible component is a hypercube, then there is no 2-dimensional restriction with exactly three feasible points, so there is no $\{00, 10, 01\}$ restriction. (\Leftarrow) Assume that S has no $\{00, 10, 01\}$ restriction. Then for all $x \in \{0, 1\}^n$ and distinct $i, j \in [n]$: if $x, x\Delta e_i, x\Delta e_j \in S$ then $x\Delta e_i\Delta e_j \in S$. Thus, by Proposition 4.4, every component of S is a hypercube. \square

So excluding $\{00, 10, 01\}$ restrictions has a propagating effect. In the same vein, resistance, which is equivalent to excluding fragile restrictions and $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}, k \geq 4$ by Theorem 1.8 (ii), entails propagations.

4.2 Propagations in resistant sets

Here we state three lemmas illustrating the different propagations running in resistant sets. Here is the first lemma:

Lemma 4.6 (Plane Propagation). *Take an integer $n \geq 1$ and a resistant set $S \subseteq \{0, 1\}^n$. If $S \cap \{x : x_n = 0\} = \emptyset$, then S is a hypercube.*

Proof. Let $S_1 \subseteq \{0, 1\}^n$ be the 1-restriction of S over coordinate n .

Claim 1. S_1 is strictly connected.

Proof of Claim. Suppose otherwise. Then by Proposition 4.3, there is an integer $k \geq 2$ such that S_1 has a $\{\mathbf{0}^k, \mathbf{1}^k\}$ restriction. Since $S \cap \{x : x_n = 0\} = \emptyset$, it follows that S has a $\{\mathbf{0}^{k+1}, \mathbf{1}^{k+1} - e_1\}$ restriction. As S is resistant, Theorem 1.8 (ii) implies that $k = 2$. However, $\{\mathbf{0}^3, \mathbf{1}^3 - e_1\}$ is fragile, so S has a fragile restriction, a contradiction to Theorem 1.8 (ii). \diamond

Claim 2. Every component of S_1 is a hypercube.

Proof of Claim. Suppose otherwise. It follows from Corollary 4.5 that S_1 has a $\{00, 10, 01\}$ restriction. As $S \cap \{x : x_n = 0\} = \emptyset$, it follows that S has a $\{000, 100, 010\}$ restriction. However, $\{000, 100, 010\}$ is fragile, a contradiction to Theorem 1.8 (ii). \diamond

Claims 1 and 2 together imply that S_1 is a hypercube, so S is a hypercube because $S \cap \{x : x_n = 0\} = \emptyset$, as required. \square

For the next lemma, let us start with the following remark illustrated in Figure 5:

Remark 4.7. *Take an integer $n \geq 1$ and a resistant set $S \subseteq \{0, 1\}^n$, where $\mathbf{0}, e_1$ are infeasible. Assume that y is a minimal feasible point such that $y_1 = 0$. Then $\{z \in S : z \leq y + e_1, z_1 = 1\} \subseteq \{y + e_1\}$.*

Proof. Suppose otherwise. Pick a minimal point z of $\{z \in S : z \leq y + e_1, z_1 = 1\}$. Our contrary assumption implies that $z \neq y + e_1$, and therefore, z is also a minimal point of S . Moreover, as e_1 is infeasible, $z \neq e_1$. Pick members $C, C' \in \text{ind}(S)$ such that $y = \chi_C$ and $z = \chi_{C'}$. Then $C \cap C' \neq \emptyset$, a contradiction as S is resistant. \square

Let us phrase this remark in a more applicable language. Take an integer $n \geq 1$ and a resistant set $S \subseteq \{0, 1\}^n$. A *valid pair* is a pair $[x, y]$ where x is infeasible, y is feasible, and $y \triangle x$ is a minimal feasible point of $S \triangle x$. If $[x, y]$ is a valid pair, we will say that x sees y . Remark 4.7 has the following immediate consequence:

Lemma 4.8 (Sight Propagation). *Take an integer $n \geq 1$, a resistant set $S \subseteq \{0, 1\}^n$ and a valid pair $[x, y]$. For a coordinate $i \in [n]$ such that $x \triangle e_i$ is infeasible, exactly one of the following statements holds:*

- (i) $y \triangle e_i$ is feasible and $[x \triangle e_i, y \triangle e_i]$ is valid,

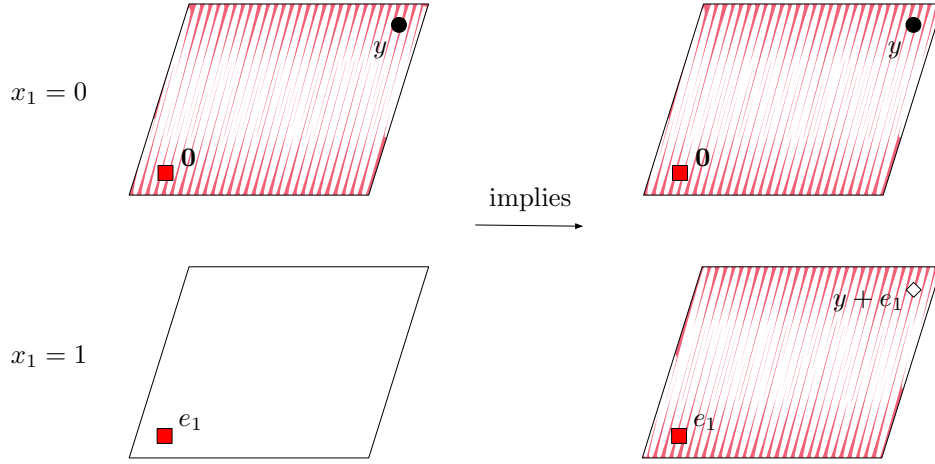


Figure 5: An illustration of Remark 4.7. Round points are feasible, the square points and the shaded region are infeasible, while the diamond point can be either.

(ii) $y \triangle e_i$ is infeasible and $[x \triangle e_i, y]$ is valid.

Proof. After a possible twisting and relabeling, if necessary, we may assume that $x = \mathbf{0}$ and $i = 1$. As $[0, y]$ is valid, y is a minimal feasible point. If $y_1 = 1$, then clearly (ii) holds and (i) does not. Otherwise, $y_1 = 0$. Then by Remark 4.7, $\{z \in S : z \leq y + e_1, z_1 = 1\}$ is either \emptyset or $\{y + e_1\}$. In the first case, (ii) holds and (i) does not, while in the second case, (i) holds and (ii) does not. \square

The Sight Propagation Lemma has a subtle implication, which leads to the third propagation lemma. Take an integer $n \geq 1$, a resistant set $S \subseteq \{0, 1\}^n$ and an infeasible point x . A *valid sequence for x* is a nonempty sequence (i_1, i_2, \dots, i_k) of (not necessarily distinct) coordinates in $[n]$ such that the points

$$x \triangle e_{i_1}, x \triangle e_{i_1} \triangle e_{i_2}, \dots, x \triangle e_{i_1} \triangle e_{i_2} \triangle \dots \triangle e_{i_k}$$

are infeasible. Take a valid pair $[x, y]$ and a valid sequence (i_1, \dots, i_k) for x . In what follows, we will define the *trajectory of $[x, y]$ along (i_1, \dots, i_k)* as a sequence (t_1, \dots, t_k) whose entries are in $\{0, 1\}$, and given the sequence, we define the *image of $[x, y]$ along (i_1, \dots, i_k)* as

$$\text{im}[x, y](i_1, \dots, i_k) := y + \sum_{j=1}^k t_j e_{i_j} \pmod{2}.$$

The sequence (t_1, \dots, t_k) is defined as follows:

- for a valid pair $[x, y]$ and a valid sequence (i) of length 1, the trajectory of $[x, y]$ along (i) is

$$T[x, y](i) := \begin{cases} (1) & \text{if } y \triangle e_i \in S \\ (0) & \text{if } y \triangle e_i \notin S, \end{cases}$$

- for a valid pair $[x, y]$ and a valid sequence (i_1, \dots, i_k) of length at least 2, the trajectory of $[x, y]$ along (i_1, \dots, i_k) is defined recursively as follows: let $y' := \text{im}[x, y](i_1, \dots, i_{k-1})$ and

$$T[x, y](i_1, \dots, i_k) := \begin{cases} T[x, y](i_1, \dots, i_{k-1}) \cup (1) & \text{if } y' \Delta e_{i_k} \in S \\ T[x, y](i_1, \dots, i_{k-1}) \cup (0) & \text{if } y' \Delta e_{i_k} \notin S. \end{cases}$$

The following is an immediate consequence of the Sight Propagation Lemma:

Remark 4.9. Take an integer $n \geq 1$, a resistant set $S \subseteq \{0, 1\}^n$, a valid pair $[x, y]$ and a valid sequence (i_1, \dots, i_k) for x . Then $\text{im}[x, y](e_{i_1}, \dots, e_{i_k})$ is feasible and is seen by $x \Delta e_{i_1} \Delta \dots \Delta e_{i_k}$.

We are now ready for the third propagation lemma:

Lemma 4.10 (Path Propagation). Take an integer $n \geq 1$, a resistant set $S \subseteq \{0, 1\}^n$ and a straight infeasible (a, b) -path P contained in $\{x : x_n = 0\}$. If $a \Delta e_n, b \Delta e_n$ are feasible, then for every vertex v of P , $v \Delta e_n$ is feasible.

Proof. If a, b are the only vertices of P , then we are clearly done. Otherwise, as P is straight and contained in $\{x : x_n = 0\}$, we may assume by Remark 4.2 that after a possible relabeling,

$$P = \left(a = \mathbf{0}, e_1, e_1 + e_2, \dots, \sum_{i=1}^k e_i = b \right),$$

where $k \in \{2, \dots, n-1\}$. Assuming that $a \Delta e_n = e_n$ and $b \Delta e_n = e_n + \sum_{i=1}^k e_i$ are feasible, we need to show that the points $e_n + \sum_{i=1}^j e_i, j \in [k-1]$ are feasible. To this end, as P is infeasible, the sequence $(1, \dots, k)$ is valid for $\mathbf{0}$. Consider the valid pair $[\mathbf{0}, e_n]$ and the valid sequence $(1, \dots, k)$. Let

$$(t_1, \dots, t_k) := T[\mathbf{0}, e_n](1, \dots, k)$$

$$y := \text{im}[\mathbf{0}, e_n](1, \dots, k) = e_n + \sum_{i=1}^k t_i e_i.$$

By Remark 4.9, y is a feasible point seen by $\mathbf{0} + \sum_{i=1}^k e_i = b$.

Claim. $y = e_n + \sum_{i=1}^k e_i$.

Proof of Claim. We know that b sees y , and clearly, b sees $b \Delta e_n = e_n + \sum_{i=1}^k e_i$ too. Suppose for a contradiction that $y \neq b \Delta e_n$. As S is resistant, the members of $\text{ind}(S \Delta b)$ are pairwise disjoint, so the supposedly distinct points $y \Delta b$ and $b \Delta e_n \Delta b = e_n$ must have disjoint supports, a contradiction as $(y \Delta b)_n = 1$. \diamond

As an immediate consequence, $t_1 = t_2 = \dots = t_k = 1$. Take a coordinate $j \in [k-1]$. Then the image of the valid pair $[\mathbf{0}, e_n]$ along the valid sequence $(1, \dots, j)$ for $\mathbf{0}$ is

$$\text{im}[\mathbf{0}, e_n](1, \dots, j) = e_n + \sum_{i=1}^j t_i e_i = e_n + \sum_{i=1}^j e_i.$$

Thus, by Remark 4.9, $e_n + \sum_{i=1}^j e_i$ is feasible, as required. \square

4.3 Global implications for resistant sets

Applying the Plane Propagation Lemma Let us prove Theorem 1.13, stating that for a set $S \subseteq \{0, 1\}^n$, (i) S is resistant and strictly polar if, and only if, (ii) in every restriction of S , either there are antipodal feasible points or the feasible points form a hypercube.

Proof of Theorem 1.13. **(i) \Rightarrow (ii):** Assume that S is resistant and strictly polar. By Remark 3.2, every restriction of S is also resistant, and by definition, every restriction of S is also strictly polar. Thus, it suffices to show that either S has antipodal points or S is a hypercube. To this end, assume that S does not have antipodal points. As S is polar, the points in S must all agree on a coordinate. The Plane Propagation Lemma now implies that S is a hypercube, as required. **(ii) \Rightarrow (i):** Assume that in every restriction of S ,

(\star) there are either antipodal feasible points or the feasible points form a hypercube.

Obviously, every restriction of S is polar, so S is strictly polar. It remains to show that S is resistant. By Theorem 1.8 (ii), it suffices to show that S has no fragile or $\{\mathbf{0}^k, \mathbf{1}^k - e_1\}, k \geq 4$ restriction, and as these restrictions do not satisfy (\star), we are done. \square

Applying the Sight Propagation Lemma The first application of this lemma is the following theorem that we will not need, but it is interesting enough that we include it here anyway:

Theorem 4.11. *Take an integer $n \geq 1$ and a nonempty resistant set $S \subseteq \{0, 1\}^n$. Then for every infeasible component K , there is a constant $m_K \in \{1 \dots, n\}$ such that every infeasible point of K sees exactly m_K feasible points.*

Proof. Take neighboring infeasible points $x, x\Delta e_i$ and let $m(x), m(x\Delta e_i) \geq 1$ be the number of feasible points $x, x\Delta e_i$ see, respectively. It suffices to show that $m(x) = m(x\Delta e_i)$. By symmetry, it actually suffices to show that $m(x\Delta e_i) \geq m(x)$. Let $y_1, \dots, y_{m(x)}$ be the feasible points x sees. By definition, $y_1\Delta x, \dots, y_{m(x)}\Delta x$ are the minimal points of $S\Delta x$, which have pairwise disjoint supports as S is resistant. In particular, $y_1, \dots, y_{m(x)}$ are pairwise at distance at least two. For each $j \in [m(x)]$, $x\Delta e_i$ sees one of $y_j, y_j\Delta e_i$ by the Sight Propagation Lemma. Thus, it follows from the distance condition that $x\Delta e_i$ sees at least $m(x)$ feasible points, so $m(x\Delta e_i) \geq m(x)$, as required. \square

For the next application, Theorem 1.14, take an integer $n \geq 3$ and a set $S \subseteq \{0, 1\}^n$. We say that S is *critically nonpolar* if it is strictly nonpolar and, for each $i \in [n]$, both the 0- and 1-restrictions of S over coordinate i have antipodal points. The Plane and Sight Propagation Lemmas have the following implication:

Proposition 4.12. *A resistant strictly nonpolar set is critically nonpolar.*

Proof. Take an integer $n \geq 3$ and a resistant strictly nonpolar set $S \subseteq \{0, 1\}^n$. Suppose for a contradiction that S is not critically nonpolar. After twisting and relabeling, if necessary, we may assume that S_1 , the 1-restriction of S over coordinate 1, does not have antipodal points. As S is strictly nonpolar, S_1 is polar so its points must

agree on a coordinate. Since it is resistant, the Plane Propagation Lemma implies that S_1 is a hypercube. After twisting and relabeling, if necessary, we may assume that

$$(\diamond) \quad S \cap \{x : x_1 = 1\} = \{x \in \{0, 1\}^n : x_1 = x_2 = \cdots = x_k = 1\}$$

for some $k \in \{2, 3, \dots, n\}$.

Claim. $S \cap \{x_1 = 0\} \subseteq \{x \in \{0, 1\}^n : x_2 = \cdots = x_k = 1\}$.

Proof of Claim. Take a point $b \in \{0, 1\}^n$ such that $b_1 = 0$ and $b \notin \{x : x_2 = \cdots = x_k = 1\}$. We need to show that b is infeasible. To this end, let a, c be the points in $\{x : x_1 = 0, x_2 = \cdots = x_k = 0\}, \{x : x_1 = 0, x_2 = \cdots = x_k = 1\}$ that otherwise agree with b , respectively. By assumption, $b \neq c$. Since (\diamond) holds and S does not have antipodal points, a is infeasible. Thus, as $[a + e_1, c + e_1]$ is a valid pair, and a is infeasible, the Sight Propagation Lemma implies that a sees one of $c, c + e_1$. As a result, all the points in $\{x : a \leq x \leq c, x \neq c\}$, including b , are infeasible, as required. \diamond

In particular, the points in S agree on coordinate 2, a contradiction as S is nonpolar. Thus, S is critically nonpolar. \square

We will need the following result:

Theorem 4.13 ([1], Theorem 3.6). *Take an integer $n \geq 3$ and a strictly nonpolar set $S \subseteq \{0, 1\}^n$. Then the following statements are equivalent:*

- (i) *cuboid(S) is minimally nonpacking,*
- (ii) *S is critically nonpolar, and every induced clutter of S has the packing property.*

We are now ready to prove Theorem 1.14, stating that a resistant set is strictly nonpolar if and only if its cuboid is an ideal minimally nonpacking clutter.

Proof of Theorem 1.14. Take an integer $n \geq 3$ and let $S \subseteq \{0, 1\}^n$ be a resistant set. Then by Corollary 1.5, cuboid(S) is ideal. (\Rightarrow) Assume that S is strictly nonpolar. By Proposition 4.12, S is critically nonpolar, and by Remark 1.11, every induced clutter of S has the packing property. Thus, Theorem 4.13 implies that cuboid(S) is minimally nonpacking, as required. (\Leftarrow) Assume that cuboid(S) is minimally nonpacking. Then by Theorem 4.13, S is critically nonpolar, so S is strictly nonpolar also, thereby finishing the proof. \square

Applying the Path Propagation Lemma Finally, let us see the following application of this lemma, which will be useful later:

Theorem 4.14. *Take an integer $n \geq 3$ and a resistant strictly nonpolar set $S \subseteq \{0, 1\}^n$. Then \overline{S} and S are not strictly connected.*

Proof. By Proposition 4.12, S is critically nonpolar. Thus, after a possible twisting, we may assume that $\mathbf{0}, \mathbf{1} - e_n \in S$. As S does not contain antipodal points, we have that $e_n, \mathbf{1} \in \overline{S}$.

Claim 1. \overline{S} is not strictly connected.

Proof of Claim. Suppose for a contradiction that \overline{S} is strictly connected. We will show that

(\star) for every infeasible point a in $\{x \in \{0, 1\}^n : x_n = 1\}$, the point $a\Delta e_n$ is feasible.

To this end, as \overline{S} is strictly connected, there exist a straight infeasible (e_n, a) -path P and a straight infeasible $(a, \mathbf{1})$ -path Q , by Proposition 4.3. Since $\text{dist}(e_n, a) + \text{dist}(a, \mathbf{1}) = \text{dist}(e_n, \mathbf{1})$, it follows from Remark 4.2 that $P \cup Q$ is a straight infeasible $(e_n, \mathbf{1})$ -path, which contains a . Thus, as $\mathbf{0}, \mathbf{1} - e_n$ are feasible, the Path Propagation Lemma implies that $a\Delta e_n$ is feasible, thereby proving (\star). (\star) implies in particular that every infeasible path is contained in either $\{x : x_n = 1\}$ or $\{x : x_n = 0\}$. Applying Proposition 4.3, we see that in fact, $\overline{S} \subseteq \{x : x_n = 1\}$. Since S does not contain antipodal points, we must have that $\overline{S} = \{x : x_n = 1\}$ and $S = \{x : x_n = 0\}$, implying in turn that S is polar, a contradiction. \diamond

Claim 2. S is not strictly connected.

Proof of Claim. Suppose for a contradiction that S is strictly connected. We will show that

(\diamond) for every feasible point b in $\{x \in \{0, 1\}^n : x_n = 0\}$, the point $b\Delta e_n$ is infeasible.

To this end, as S is strictly connected, there exist a straight feasible $(\mathbf{0}, b)$ -path P as well as a straight feasible $(b, \mathbf{1} - e_n)$ -path Q , by Proposition 4.3. Since $\text{dist}(\mathbf{0}, b) + \text{dist}(b, \mathbf{1} - e_n) = \text{dist}(\mathbf{0}, \mathbf{1} - e_n)$, it follows from Remark 4.2 that $P \cup Q$ is a straight feasible $(\mathbf{0}, \mathbf{1} - e_n)$ -path, which contains b . Since S does not have antipodal points, it follows that $(P \cup Q)\Delta \mathbf{1}$ is a straight infeasible $(e_n, \mathbf{1})$ -path.⁶ The Path Propagation Lemma now implies that $(P \cup Q)\Delta \mathbf{1}\Delta e_n$ is a straight feasible $(\mathbf{0}, \mathbf{1} - e_n)$ -path. Once again, as S does not have antipodal points, we get that $(P \cup Q)\Delta e_n$ is a straight infeasible $(e_n, \mathbf{1})$ -path, implying in particular that $b\Delta e_n$ is infeasible, thereby proving (\diamond). (\diamond) implies in particular that every feasible path is contained in either $\{x : x_n = 0\}$ or $\{x : x_n = 1\}$. Applying Proposition 4.3, we see that in fact, $S \subseteq \{x : x_n = 0\}$, implying in turn that S is polar, a contradiction. \diamond

Claims 1 and 2 finish the proof of Theorem 4.14. \square

As an immediate consequence,

Corollary 4.15. *Take an integer $n \geq 1$ and a resistant set $S \subseteq \{0, 1\}^n$. If \overline{S} or S is strictly connected, then S is strictly polar.*

⁶If P is an (a, b) -path, then $P\Delta x$ denotes the $(a\Delta x, b\Delta x)$ -path whose vertices are the vertices of P twisted by x .

5 Straight circuits

Take an integer $n \geq 2$. Let C be a circuit of G_n whose vertices, denoted $V(C)$, are v_0, v_1, \dots, v_k in clockwise order. We will represent C as the sequence $(v_0, v_1, \dots, v_k, v_0)$. Take a point $v \in \{0, 1\}^n$, an integer $\ell \in \{2, \dots, n\}$, and distinct coordinates $i_1, \dots, i_\ell \in [n]$. Denote by $(v : i_1, i_2, \dots, i_\ell)$ the circuit

$$(v_0, v_1, \dots, v_\ell, \dots, v_{2\ell-1}, v_{2\ell} = v_0)$$

where $v_0 = v$ and $v_j = v_{j-1} \Delta e_{i_j}$ and $v_{\ell+j} = v_{\ell+j-1} \Delta e_{i_j}$ for each $j \in [\ell]$. We will refer to $(v : i_1, i_2, \dots, i_\ell)$ as a *straight circuit*. (Notice that any point of the straight circuit can be a starting point.)

The *length of a circuit* is the number of edges it has. Take a set $S \subseteq \{0, 1\}^n$. We refer to every circuit of $G_n[S]$ as a *feasible circuit* and to every circuit of $G_n[\bar{S}]$ as an *infeasible circuit*. The purpose of this section is to prove the following statement:

Take an integer $n \geq 4$ and a resistant set $S \subseteq \{0, 1\}^n$ that is nonpolar. Assume that there is a straight infeasible circuit K of length $2(n-1)$ contained in $\{x : x_n = 0\}$ such that $V(K \Delta e_n) \subseteq S$. Then S has one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction.⁷

This tool is crucial for proving the main result of the paper, Theorem 1.16. To prove this statement, let us start with the following lemma that is widely referenced throughout this section:

Lemma 5.1 (Straight Circuit). *Take an integer $n \geq 4$ and a resistant set $S \subseteq \{0, 1\}^n$ without antipodal points. Let K be a straight infeasible circuit of length $2(n-1)$ contained in $\{x : x_n = 0\}$ such that $V(K \Delta e_n) \subseteq S$. Then for a vertex $v \in \{x : x_n = 0\} - V(K)$ that is adjacent to a vertex of K , either*

$$\{v, \mathbf{1} - v \Delta e_n\} \subseteq \bar{S} \quad \text{and} \quad \{v \Delta e_n, \mathbf{1} - v\} \subseteq S$$

or

$$\{v, \mathbf{1} - v \Delta e_n\} \subseteq S \quad \text{and} \quad \{v \Delta e_n, \mathbf{1} - v\} \subseteq \bar{S}.$$

Proof. After a possible relabeling and twisting, we may assume that

$$K = (\mathbf{0} : 1, 2, \dots, n-1) = (v_0, v_1, \dots, v_{n-1}, \dots, v_{2n-3}, v_{2n-2} = v_0),$$

where $v_0 = \mathbf{0}$ and $v_j = v_{j-1} \Delta e_j$ and $v_{n-1+j} = v_{n-1+j-1} \Delta e_j$ for each $j \in [n-1]$.

Claim. *Take a vertex $w \in \{x : x_n = 0\} - V(K)$ that is adjacent to a vertex of K . If $w \in \bar{S}$ then $w \Delta e_n \in S$.*

Proof of Claim. By the symmetry between the vertices of K , we may assume that w is adjacent to v_0 , that is, $w = v_0 \Delta e_i$ for some $i \in [n-1] - \{1, n-1\}$. Let

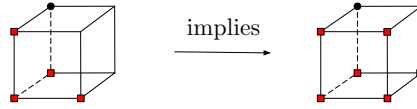
$$P := (v_{n-1+i}, v_{n-1+i+1}, \dots, v_{2n-2} = v_0) \quad \text{and} \quad Q := (v_0, v_1, \dots, v_{i-1}) \quad \text{and} \quad R := P \cup Q.$$

⁷If C is a circuit, then $C \Delta x$ denotes the circuit whose vertices are the vertices of C twisted by x .

Notice that R is a straight subpath of the infeasible circuit K . The Path Propagation Lemma implies that the feasible points in $R\Delta e_i$ should form a path. Thus, since $v_0\Delta e_i = w \in \bar{S}$, it follows that either $V(P\Delta e_i) \subseteq \bar{S}$ or $V(Q\Delta e_i) \subseteq \bar{S}$. By symmetry, we may assume that $V(P\Delta e_i) \subseteq \bar{S}$. Consider now the straight infeasible path $P' := (v_0) \cup [P\Delta e_i]$ whose ends are v_0 and $v_{n-1+i}\Delta e_i = v_{n-1+i-1}$. Since $\{v_0\Delta e_n, v_{n-1+i-1}\Delta e_n\} \subseteq V(K\Delta e_n) \subseteq S$, it follows from the Path Propagation Lemma that $P'\Delta e_n$ is a feasible path. In particular, we have that $v_0\Delta e_i\Delta e_n = w\Delta e_n \in S$. \diamond

Now take a vertex $v \in \{x : x_n = 0\} - V(K)$ that is adjacent to a vertex of K . Assume in the first case that $v \in \bar{S}$. By the claim, $v\Delta e_n \in S$. Since S does not contain antipodal points, we get that $\mathbf{1} - v\Delta e_n \in \bar{S}$. Since $\mathbf{1} - v\Delta e_n$ is also adjacent to a vertex of K , and is not a vertex of K , it follows from the claim that $\mathbf{1} - v \in S$. Assume in the remaining case that $v \in S$. As S does not contain antipodal points, $\mathbf{1} - v \in \bar{S}$, so by the claim, we have $\mathbf{1} - v\Delta e_n \in S$, so its antipodal point $v\Delta e_n$ belongs to \bar{S} . This finishes the proof of the lemma. \square

Before moving on, we should point out that the results in this section will make heavy use of the Sight Propagation Lemma, most often applied as illustrated in the following figure,



and in most cases, we will leave it to the reader to identify the 3-dimensional cube where the Sight Propagation Lemma is being applied.

5.1 When there are no $R_{1,1}, R_5$ restrictions

We will need the following technical lemma:

Lemma 5.2. *Take an integer $n \geq 6$ and a resistant set $S \subseteq \{0, 1\}^n$ without antipodal points and without an $R_{1,1}, R_5$ restriction. Suppose $K := (\mathbf{0} : 1, 2, 3, 4, 5, \dots, n-1)$ is a straight infeasible circuit, $V(K\Delta e_n) \subseteq S$, and $\{e_2, e_2 + e_3, e_3, e_1 + e_3\} \subseteq S$. Then, for each $i \in [n-4]$,*

$$K_i := (\mathbf{0} : 4, \dots, 3+i, 1, 2, 3, 3+i+1, \dots, n-1)$$

is a straight infeasible circuit, $V(K_i\Delta e_n) \subseteq S$ and $\{e_2, e_2 + e_3, e_3, e_1 + e_3\}\Delta e_4\Delta \dots \Delta e_{3+i} \subseteq S$.

Proof. We proceed by induction on $i \geq 1$. Let us first prove the base case $i = 1$, which we restate as follows:

(\star) Take an integer $n \geq 6$ and a resistant set $S \subseteq \{0, 1\}^n$ without antipodal points and without an $R_{1,1}, R_5$ restriction. Suppose $K := (\mathbf{0} : 1, 2, 3, 4, 5, \dots, n-1)$ is a straight infeasible circuit, $V(K\Delta e_n) \subseteq S$, and $\{e_2, e_2 + e_3, e_3, e_1 + e_3\} \subseteq S$. Then, $K' := (\mathbf{0} : 4, 1, 2, 3, 5, \dots, n-1)$ is a straight infeasible circuit, $V(K'\Delta e_n) \subseteq S$ and $\{e_2, e_2 + e_3, e_3, e_1 + e_3\}\Delta e_4 \subseteq S$.

Define $P_0, P_1, Q_0, Q_1 \subseteq \{0, 1\}^4$ as follows: P_0 (resp. P_1) is obtained after 0-restricting coordinates $5, \dots, n-1$ and 0-restricting coordinate n (resp. 1-restricting coordinate n), and Q_0 (resp. Q_1) is obtained after 1-restricting coordinates $5, \dots, n-1$ and 0-restricting coordinate n (resp. 1-restricting coordinate n). Since $V(K) \subseteq \bar{S}$ and $V(K \triangle e_n) \subseteq S$, it follows that

$$\begin{aligned} \{0000, 1000, 1100, 1110, 1111\} &\subseteq \bar{P}_0 & \{0000, 1000, 1100, 1110, 1111\} &\subseteq P_1 \\ \{1111, 0111, 0011, 0001, 0000\} &\subseteq \bar{Q}_0 & \{1111, 0111, 0011, 0001, 0000\} &\subseteq Q_1. \end{aligned}$$

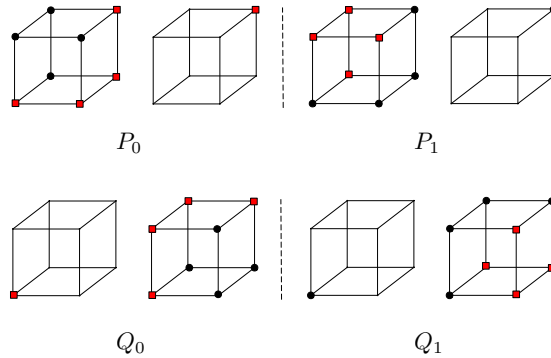
By assumption, we also know that

$$\{0100, 0110, 0010, 1010\} \subseteq P_0.$$

In $\{0, 1\}^n$, each one of these points belongs to \bar{K} and is adjacent to a vertex of K , so by the Straight Circuit Lemma,

$$\begin{aligned} \{0100, 0110, 0010, 1010\} &\subseteq \bar{P}_1 \\ \{1011, 1001, 1101, 0101\} &\subseteq Q_0 \\ \{1011, 1001, 1101, 0101\} &\subseteq \bar{Q}_1. \end{aligned}$$

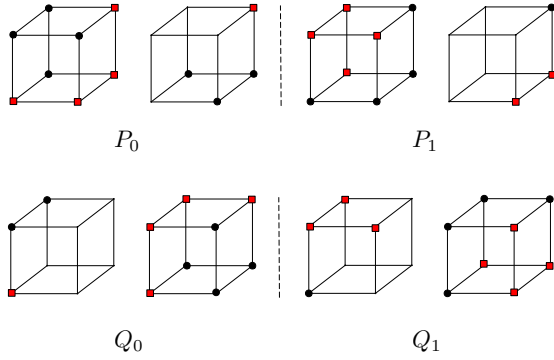
See the following figure illustrating the inclusions listed so far:



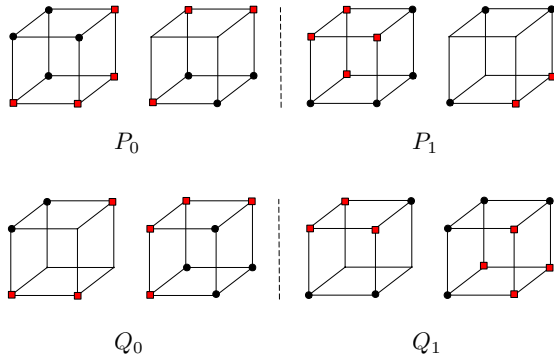
In the following claim, we will take advantage of the assumption that S has no $R_{1,1}, R_5$ restriction.

Claim 1. $\{0001, 1001, 1101\} \subseteq \bar{P}_0 \cap P_1$ and $\{1110, 0110, 0010\} \subseteq \bar{Q}_0 \cap Q_1$.

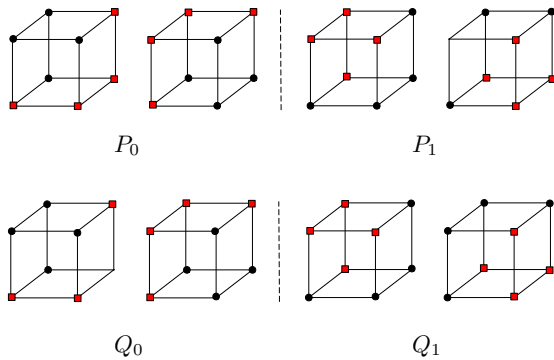
Proof of Claim. We will first show that $1101 \in \bar{P}_0$. Suppose for a contradiction that $1101 \in P_0$. Since S is resistant, it follows from the Sight Propagation Lemma that $1001 \in P_0$. As the vertices in $\{0, 1\}^n$ corresponding to $1101, 1001$ are in \bar{K} and adjacent to vertices of K , the Straight Circuit Lemma implies that $\{1101, 1001\} \subseteq \bar{P}_1$ and $\{0010, 0110\} \subseteq Q_0 \cap \bar{Q}_1$. Since the restriction of $(P_0 \times \{0\}) \cup (P_1 \times \{1\})$ obtained after 1-restricting coordinate 2 and 0-restricting coordinate 3 is neither P_3 nor $R_{1,1}$, it follows that $0101 \in P_0$. As S does not contain antipodal points, we get that $1010 \in \bar{Q}_1$:



Since the 0-restriction of Q_1 over coordinate 2 is resistant, it follows that $1000 \in Q_1$, so by the Straight Circuit Lemma, $1000 \in \overline{Q_0}$ and $0111 \in \overline{P_0} \cap P_1$. As the 1-restriction of Q_1 over coordinate 3 is resistant, we have $1110 \in Q_1$, and so by the Straight Circuit Lemma, $1110 \in \overline{Q_0}$ and $0001 \in \overline{P_0} \cap P_1$:



Since the 1-restriction of P_0 over coordinate 3 is resistant, we get that $1011 \in P_0$, and by the Straight Circuit Lemma, $1011 \in \overline{P_1}$ and $0100 \in Q_0 \cap \overline{Q_1}$. As the restriction of $(Q_0 \times \{0\}) \cup (Q_1 \times \{1\})$ obtained after 1-restricting coordinate 1 and 0-restricting coordinate 4 is resistant and has no $R_{1,1}$ restriction, it follows that $1100 \in Q_1$ and $1010 \in Q_0$. Since S does not have antipodal points, we get that $0011 \in \overline{P_0}$ and $0101 \in \overline{P_1}$:



Since the 0-restriction of P_1 over coordinate 2 is resistant, it follows that $0011 \in P_1$, implying in turn that $(P_0 \times \{0\}) \cup (P_1 \times \{1\}) \cong R_5$, so S has an R_5 restriction, a contradiction. Thus, $1101 \in \overline{P_0}$.

It follows from the Sight Propagation Lemma that $\{1001, 0001\} \subseteq \overline{P_0}$. By the Straight Circuit Lemma, $\{0001, 1001, 1101\} \subseteq P_1$ and $\{1110, 0110, 0010\} \subseteq \overline{Q_0} \cap Q_1$, as claimed. \diamond

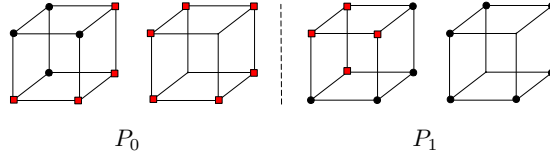
Recall that $K' = (\mathbf{0} : 4, 1, 2, 3, 5, \dots, n-1)$. Notice that by Claim 1, K' is a straight infeasible circuit such that $V(K' \triangle e_n) \subseteq S$. In the following claim, we will apply the Straight Circuit Lemma to the straight infeasible circuit K' .

Claim 2. $\{0101, 0111, 0011, 1011\} \subseteq P_0$.

Proof of Claim. Since P_1 is resistant, it follows from the Sight Propagation Lemma that either

$$\{0011, 0111\} \subseteq P_1 \quad \text{or} \quad \{0011, 0111\} \subseteq \overline{P_1}.$$

We claim that $\{0011, 0111\} \subseteq \overline{P_1}$. Suppose for a contradiction that $\{0011, 0111\} \subseteq P_1$. After applying the Straight Circuit Lemma to K' , we get that $\{0011, 0111\} \subseteq \overline{P_0}$:



Since P_0 is resistant, it follows from Theorem 1.8 (ii) that P_0 has no fragile restriction. However, either its 0-restriction over coordinate 2 or its 1-restriction over coordinate 3 is fragile, a contradiction. Thus, $\{0011, 0111\} \subseteq \overline{P_1}$. After applying the Straight Circuit Lemma to K' , we get that $\{0011, 0111\} \subseteq P_0$. As P_0 is resistant, it follows that $\{0101, 1011\} \subseteq P_0$, as required. \diamond

By Claim 2, $\{e_2, e_2 + e_3, e_3, e_1 + e_3\} \triangle e_4 \subseteq S$. This proves (\star) and in turn the base case $i = 1$. For the induction step, assume that $i \geq 2$. Then an application of (\star) to K_{i-1} , instead of K , implies that $K_i := (\mathbf{0} : 4, \dots, 3+i, 1, 2, 3, 3+i+1, \dots, n-1)$ is a straight infeasible circuit, $V(K_i \triangle e_n) \subseteq S$, and $\{e_2, e_2 + e_3, e_3, e_1 + e_3\} \triangle e_4 \triangle \dots \triangle e_{3+i} \subseteq S$, thereby completing the induction step. \square

Let $D_3 := \{000, 100, 110, 111\} \subseteq \{0, 1\}^3$. Using the preceding lemma, we prove the following:

Proposition 5.3. *Take an integer $n \geq 5$ and a resistant set $S \subseteq \{0, 1\}^n$ without antipodal points and without an $R_{1,1}, R_5$ restriction. Let K be a straight infeasible circuit of length $2(n-1)$ contained in $\{x : x_n = 0\}$ such that $V(K \triangle e_n) \subseteq S$. Then S does not have a D_3 restriction whose infeasible points all belong to K .*

Proof. After a possible relabeling and twisting, we may assume that $K = (\mathbf{0} : 1, 2, \dots, n-1)$. Suppose for a contradiction that S has a D_3 restriction whose infeasible points all belong to K . By symmetry, we may assume that the D_3 restriction is obtained after 0-restricting coordinates $4, \dots, n$, that is, $\{e_2, e_2 + e_3, e_3, e_1 + e_3\} \subseteq S$. Assume in the first case that $n \geq 6$. It then follows from Lemma 5.2 that for each $i \in [n-4]$, $K_i :=$

$(\mathbf{0} : 4, \dots, 3+i, 1, 2, 3, 3+i+1, \dots, n-1)$ is a straight infeasible circuit, $V(K_i \Delta e_n) \subseteq S$, and $\{e_2, e_2 + e_3, e_3, e_1 + e_3\} \Delta e_4 \Delta \dots \Delta e_{3+i} \subseteq S$. In particular, setting $i = n-4$, we get that

$$e_3 \Delta e_4 \Delta e_5 \Delta \dots \Delta e_{n-1} \in S.$$

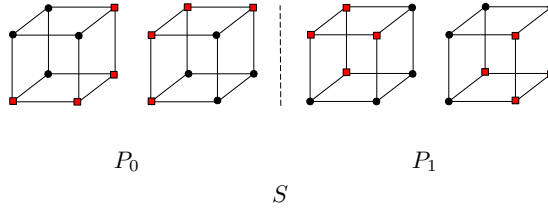
However, $e_3 \Delta e_4 \Delta e_5 \Delta \dots \Delta e_{n-1} \in K \subseteq \overline{S}$, a contradiction. Assume in the remaining case that $n = 5$. Let $P_0 \subseteq \{0, 1\}^4$ (resp. $P_1 \subseteq \{0, 1\}^4$) be the 0-restriction (resp. 1-restriction) of S over coordinate 5. Since $V(K) \subseteq \overline{S}$ and $V(K \Delta e_n) \subseteq S$, it follows that

$$\begin{aligned} \{0000, 1000, 1100, 1110, 1111, 0111, 0011, 0001\} &\subseteq \overline{P_0} \\ \{0000, 1000, 1100, 1110, 1111, 0111, 0011, 0001\} &\subseteq P_1. \end{aligned}$$

As the 0-restriction of P_0 over coordinate 4 yields D_3 , we also know that $\{0100, 0110, 0010, 1010\} \subseteq P_0$. In $\{0, 1\}^5$, each one of these points belongs to \overline{K} and is adjacent to a vertex of K , so by the Straight Circuit Lemma,

$$\begin{aligned} \{0100, 0110, 0010, 1010\} &\subseteq \overline{P_1} \\ \{1011, 1001, 1101, 0101\} &\subseteq P_0 \\ \{1011, 1001, 1101, 0101\} &\subseteq \overline{P_1}. \end{aligned}$$

The points given above determine that $S \cong R_5$,



a contradiction. □

5.2 Finding $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ restrictions

For an integer $n \geq 3$, we will need the following property defined on the points x in $\{0, 1\}^n$:

$$(\diamond) x \text{ is feasible} \Leftrightarrow \mathbf{1} - x \Delta e_n \text{ is feasible} \Leftrightarrow x \Delta e_n \text{ is infeasible} \Leftrightarrow \mathbf{1} - x \text{ is infeasible}.$$

Lemma 5.4. *Take an integer $n \geq 5$ and a resistant set $S \subseteq \{0, 1\}^n$ that does not have antipodal points, and let $K := (\mathbf{0} : 1, 2, \dots, n-1)$ be a straight infeasible circuit such that $V(K \Delta e_n) \subseteq S$. Suppose that $\ell \in \{2, \dots, n-3\}$ is an integer such that the points in $\{x : x_{\ell+1} = \dots = x_n = 0\}$ satisfy (\diamond) and the feasible points in there form a hypercube. Then one of the following statements hold:*

- *S has one of $\{R_{k,1} : 1 \leq k \leq \ell\} \cup \{R_5\}$ as a restriction, or*
- *the points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ satisfy (\diamond) and the feasible points in there form a hypercube.*

Proof. Let us proceed by induction on $\ell \geq 2$.

For the base case, assume that $\ell = 2$. Notice that every point in $\{x : x_4 = \dots = x_n = 0\}$ either belongs to K or is adjacent to a vertex of K . It therefore follows from the Straight Circuit Lemma that every point in $\{x : x_4 = \dots = x_n = 0\}$ satisfies (\diamond) . Suppose that the feasible points in $\{x : x_4 = \dots = x_n = 0\}$ do not form a hypercube. Let $H \subseteq \{0, 1\}^4$ be the 0-restriction of S over coordinates $4, \dots, n-1$. Since $V(K) \subseteq \overline{S}$ and $V(K \triangle e_n) \subseteq S$, we see that $\{0000, 1000, 1100, 1110\} \subseteq \overline{H}$ and $\{0001, 1001, 1101, 1111\} \subseteq H$. As the 0-restriction of H over the last coordinate is not a hypercube, one of the following inclusions must hold:

- $\{0100, 1010\} \subseteq H$: By (\diamond) , $\{0101, 1011\} \subseteq \overline{H}$. If $|\{0010, 0110\} \cap H| = 0$, then by (\diamond) , $H \cong R_{2,1}$, so S has an $R_{2,1}$ restriction. If $|\{0010, 0110\} \cap H| = 1$, then by (\diamond) , H , and therefore S , has an $R_{1,1}$ restriction. Otherwise, when $|\{0010, 0110\} \cap H| = 2$, then $H \cong D_3$ and so S has a D_3 restriction whose infeasible points all belong to K , so by Proposition 5.3, S has one of $R_{1,1}, R_5$ as a restriction.
- $\{0110, 1010\} \subseteq H$: Since H is resistant, it follows that $0100 \in H$, so by the preceding case, S has one of $R_{1,1}, R_{2,1}, R_5$ as a restriction.
- $\{0100, 0010\} \subseteq H$: Since H is resistant, it follows that $1010 \in H$, so by the first case, S has one of $R_{1,1}, R_{2,1}, R_5$ as a restriction.

In each case, we see that S has one of $R_{1,1}, R_{2,1}, R_5$ as a restriction, thereby proving the base case $\ell = 2$.

For the induction step, assume that $\ell \geq 3$. Then $n \geq 6$. Let $S' := S \cap \{x : x_{\ell+1} = \dots = x_n = 0\}$. By assumption, S' is a (possibly empty) hypercube, which excludes the points $\mathbf{0}, \sum_{i=1}^{\ell} e_i$ as these two points belong to the infeasible circuit K . Since S' is a hypercube and the points in $\{x : x_{\ell+1} = \dots = x_n = 0\}$ satisfy (\diamond) ,

- (1) every infeasible point of $\{x : x_{\ell+1} = \dots = x_n = 0\}$ appears on a straight infeasible circuit $K' := (\mathbf{0} : i_1, \dots, i_{\ell}, \ell + 1, \dots, n - 1)$ such that $V(K' \triangle e_n) \subseteq S$, where i_1, \dots, i_{ℓ} is some permutation of $1, \dots, \ell$.

We will use (1) throughout the proof to reroute the circuit K . Notice that together with the Straight Circuit Lemma, (1) implies that

- (2) every point of $\{x : x_{\ell+2} = \dots = x_n = 0\}$ adjacent to an infeasible point of $\{x : x_{\ell+1} = \dots = x_n = 0\}$ satisfies (\diamond) .

As a result, if the feasible points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ form a hypercube, then every infeasible point in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ satisfies (\diamond) , and so every infeasible point of $\{x : x_{\ell+2} = \dots = x_n = 0\}$ appears on a straight infeasible circuit $K' := (\mathbf{0} : i_1, \dots, i_{\ell+1}, \ell + 2, \dots, n - 1)$ such that $V(K' \triangle e_n) \subseteq S$, where $i_1, \dots, i_{\ell+1}$ is some permutation of $1, \dots, \ell + 1$. Thus, by the Straight Circuit Lemma,

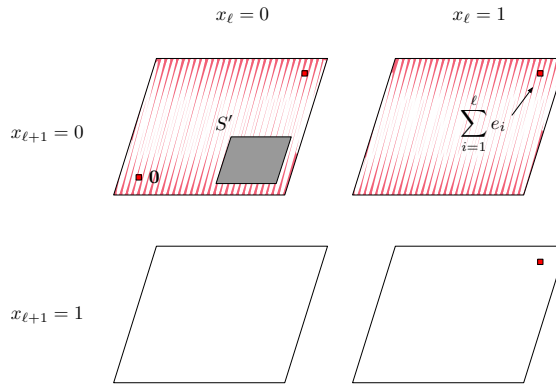
- (3) if the feasible points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ form a hypercube, then every infeasible point in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ satisfies (\diamond) .

Suppose that S has none of $\{R_{k,1} : 1 \leq k \leq \ell\} \cup \{R_5\}$ as a restriction. By (3), it suffices to show that the feasible points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ form a hypercube. As $\mathbf{0}, \sum_{i=1}^{\ell} e_i \notin S'$, it follows that S' is a hypercube of dimension at most $\ell - 2$. There are five cases:

- (i) $S' = \emptyset$,
- (ii) S' is nonempty, of dimension at most $\ell - 3$, and has no vertex adjacent to $\sum_{i=1}^{\ell} e_i$,
- (iii) S' is nonempty, of dimension at most $\ell - 3$, and has a vertex adjacent to $\sum_{i=1}^{\ell} e_i$,
- (iv) S' is of dimension $\ell - 2$ and $\ell = 3$,
- (v) S' is of dimension $\ell - 2$ and $\ell \geq 4$.

(i) In this case, it follows from the Plane Propagation Lemma that the feasible points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ form a hypercube, thereby completing the induction step.

(ii) In this case, after possibly relabeling coordinates $1, \dots, \ell$ and rerouting K according to (1), we may assume that $S' \subseteq \{x \in \{0, 1\}^n : x_{\ell} = 0\}$ while K remains as $(\mathbf{0} : 1, \dots, n - 1)$. Consider the following illustration of $\{x : x_{\ell+2} = \dots = x_n = 0\}$:



The filled-in parallelogram shows the feasible points of S' , while the shaded area and the square vertices indicate infeasible points. As $S' \neq \emptyset$, the infeasible point $\sum_{i=1}^{\ell} e_i$ sees a feasible point in S' , so by the Sight Propagation Lemma, $\sum_{i=1}^{\ell+1} e_i$ sees a feasible point in $S' \cup (S' \Delta e_{\ell+1})$. In particular,

- (4) $S' \Delta e_{\ell+1} \Delta e_{\ell}$ contains an infeasible point,

and $\sum_{i=1}^{\ell+1} e_i - e_{\ell}$ is infeasible, and by the Straight Circuit Lemma, $\sum_{i=1}^{\ell+1} e_i - e_{\ell}$ satisfies (\diamond) . Consider now the straight infeasible circuit

$$K' := (\mathbf{0} : 1, \dots, \ell - 1, \ell + 1, \ell, \dots, n - 1)$$

such that $V(K' \Delta e_n) \subseteq S$. Let us apply the induction hypothesis to K' given that the points in $\{x : x_{\ell} = x_{\ell+1} = x_{\ell+2} = \dots = x_n = 0\}$ satisfy (\diamond) and its feasible points form a hypercube. The induction hypothesis implies that the points in $\{x : x_{\ell} = x_{\ell+2} = \dots = x_n = 0\}$ also satisfy (\diamond) and its feasible points form a

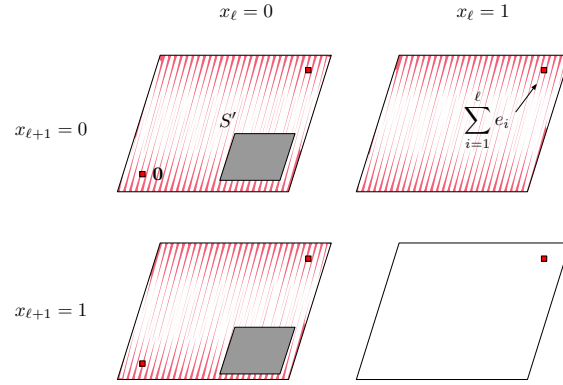
hypercube. In particular, by (2), the points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ all satisfy (\diamond) . Moreover, as $S' \neq \emptyset$, $S \cap \{x : x_\ell = x_{\ell+2} = \dots = x_n = 0\}$ is either S' or $S' \cup (S' \Delta e_{\ell+1})$.

Assume in the first case that $S \cap \{x : x_\ell = x_{\ell+2} = \dots = x_n = 0\} = S'$. We then must have that

$$S \cap \{x : x_{\ell+2} = \dots = x_n = 0\} = S'.$$

Suppose not. Pick the closest pair of feasible vertices a, b such that $a \in S'$ and $b \in \{x : x_{\ell+2} = \dots = x_n = 0\} - S'$. Since the points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ satisfy (\diamond) , it follows that the restriction of S containing $a, b \Delta e_n$ as antipodal points is one of $\{R_{k,1} : 1 \leq k \leq \ell\}$ as a restriction, a contradiction. Thus, the equation above holds, implying in turn that the feasible points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ form a hypercube, thereby completing the induction step.

Assume in the remaining case that $S \cap \{x : x_\ell = x_{\ell+2} = \dots = x_n = 0\} = S' \cup (S' \Delta e_{\ell+1})$:



Consider the straight infeasible circuit

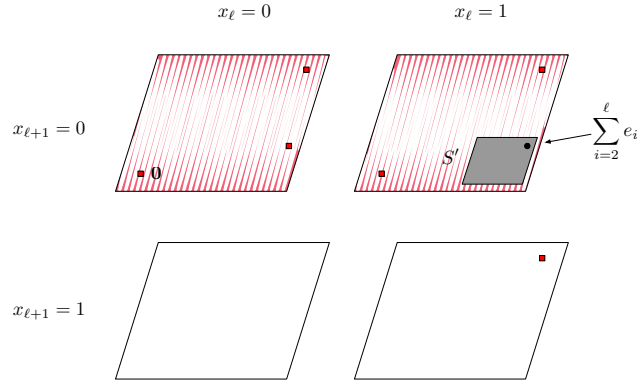
$$K'' := (e_{\ell+1} : 1, \dots, \ell, \ell + 2, \dots, n - 1, \ell + 1)$$

such that $V(K'' \Delta e_n) \subseteq S$. Let us apply the induction hypothesis to K'' given that the points in $\{x : x_\ell = x_{\ell+2} = \dots = x_n = 0, x_{\ell+1} = 1\}$ satisfy (\diamond) and its feasible points form a hypercube. The induction hypothesis implies that the feasible points in $\{x : x_{\ell+2} = \dots = x_n = 0, x_{\ell+1} = 1\}$ form a hypercube. That is, $S \cap \{x : x_{\ell+2} = \dots = x_n = 0, x_{\ell+1} = 1\}$ is either $S' \Delta e_{\ell+1}$ or $(S' \Delta e_{\ell+1}) \cup (S' \Delta e_{\ell+1} \Delta e_\ell)$. However, the latter is not possible by (4), so $S \cap \{x : x_{\ell+2} = \dots = x_n = 0, x_{\ell+1} = 1\} = S' \Delta e_{\ell+1}$ and

$$S \cap \{x : x_{\ell+2} = \dots = x_n = 0\} = S' \cup (S' \Delta e_{\ell+1}).$$

Thus, the feasible points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ form a hypercube, thereby completing the induction step.

(iii) In this case, as S' has dimension at most $\ell - 3$, it cannot have a vertex adjacent to $\mathbf{0}$. So, after possibly relabeling coordinates $1, \dots, \ell$ and rerouting K according to (1), we may assume that $S' \subseteq \{x : x_\ell = 1, x_1 = 0, x_2 = 1\}$ and $\sum_{i=2}^\ell e_i \in S'$ while K remains as $(\mathbf{0} : 1, \dots, n - 1)$:



Consider the straight circuit

$$K' := (e_\ell : 1, \dots, \ell - 1, \ell + 1, \dots, n - 1, \ell).$$

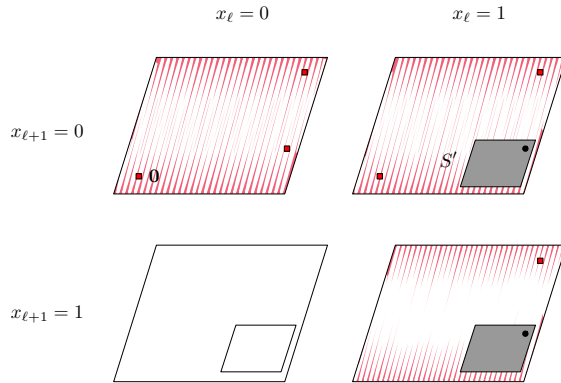
Since e_ℓ is infeasible and satisfies (\diamond) by (2), it follows that K' is infeasible and $K' \triangle e_n$ is feasible. Let us apply the induction hypothesis to K' , given that the points in $\{x : x_{\ell+1} = x_{\ell+2} = \dots = x_n = 0, x_\ell = 1\}$ satisfy (\diamond) and its feasible points form a hypercube. The induction hypothesis implies that the points in $\{x : x_{\ell+2} = \dots = x_n = 0, x_\ell = 1\}$ satisfy (\diamond) and its feasible points form a hypercube. Together with (2), this implies that all the points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ satisfy (\diamond) .

Assume in the first case that $S \cap \{x : x_{\ell+2} = \dots = x_n = 0, x_\ell = 1\} = S'$. Then we must have that

$$S \cap \{x : x_{\ell+2} = \dots = x_n = 0\} = S'.$$

Suppose otherwise. Pick a closest pair of feasible points a, b such that $a \in S'$ and $b \in \{x : x_{\ell+2} = \dots = x_n = 0\} - S'$. As the points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ satisfy (\diamond) , it follows that the restriction of S containing $a, b \triangle e_n$ as antipodal points is one of $\{R_{k,1} : 1 \leq k \leq \ell\}$, a contradiction. Thus, $S \cap \{x : x_{\ell+2} = \dots = x_n = 0\} = S'$. So the feasible points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ form a hypercube, thereby completing the induction step.

Assume in the remaining case that $S \cap \{x : x_{\ell+2} = \dots = x_n = 0, x_\ell = 1\} = S' \cup (S' \triangle e_{\ell+1})$:



We claim that all the points in $S' \Delta e_\ell \Delta e_{\ell+1}$ are infeasible. Suppose for a contradiction that, for some $x \in S'$, $x \Delta e_\ell \Delta e_{\ell+1} \in S$. Recall that $S' \subseteq \{x : x_1 = 0, x_2 = 1\}$. For $i \in \{1, 2\}$, consider the 3-dimensional cube $H_i \subseteq \{0, 1\}^3$ containing $x \Delta e_\ell, x \Delta e_{\ell+1}, x \Delta e_i$. Notice that for $i \in \{1, 2\}$,

$$\{x, x \Delta e_{\ell+1}, x \Delta e_\ell \Delta e_{\ell+1}\} \subseteq S \quad \text{and} \quad \{x \Delta e_i, x \Delta e_\ell, x \Delta e_i \Delta e_\ell, x \Delta e_i \Delta e_{\ell+1}\} \subseteq \bar{S}.$$

Thus, since H_1, H_2 are not fragile by Theorem 1.8 (ii), it follows that $x \Delta e_1 \Delta e_\ell \Delta e_{\ell+1}, x \Delta e_2 \Delta e_\ell \Delta e_{\ell+1} \in S$. To summarize, setting $y := x \Delta e_{\ell+1}$, $\{y \Delta e_\ell, y \Delta e_1 \Delta e_\ell, y \Delta e_2 \Delta e_\ell, y\} \subseteq S$. Moreover, $\{y \Delta e_1, y \Delta e_2, y \Delta e_1 \Delta e_2\} \subseteq \bar{S}$. As a result, since S does not contain antipodal points and the points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ satisfy (\diamond) , it follows that the 3-dimensional restriction of S containing $\{y \Delta e_1, y \Delta e_2, y \Delta e_\ell\} \Delta \mathbf{1}$ is fragile, a contradiction to Theorem 1.8 (ii). Thus, all the points in $S' \Delta e_\ell \Delta e_{\ell+1}$ are infeasible.

We next claim that

$$S \cap \{x : x_{\ell+2} = \dots = x_n = 0\} = S' \cup (S' \Delta e_{\ell+1}).$$

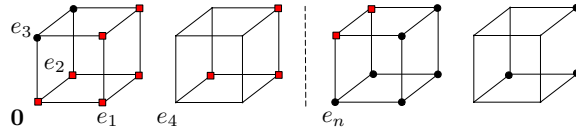
Suppose otherwise. Pick the closest pair of feasible points a, b such that $a \in S' \Delta e_{\ell+1}$ and $b \in \{x : x_{\ell+2} = \dots = x_n = 0\} - [S' \cup (S' \Delta e_{\ell+1})]$. Since all the points in $S' \Delta e_\ell \Delta e_{\ell+1}$ are infeasible, it follows that $\text{dist}(a, b) \geq 2$. Consider now the restriction of S containing $a, b \Delta e_n$ as antipodal points; because all the points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ satisfy (\diamond) , this restriction is one of $\{R_{k,1} : 1 \leq k \leq \ell - 1\}$, a contradiction. Thus, $S \cap \{x : x_{\ell+2} = \dots = x_n = 0\} = S' \cup (S' \Delta e_{\ell+1})$, implying in particular that the feasible points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ form a hypercube, thereby completing the induction step.

(iv) After possibly relabeling coordinates 1, 2, 3 and rerouting K according to (1), we may assume that $S' = \{e_3, e_2 + e_3\}$ while K remains as $(\mathbf{0} : 1, \dots, n - 1)$. By the Straight Circuit Lemma, $\bar{S} \cap \{x : x_4 = \dots = x_{n-1} = 0, x_n = 1\} = \{e_3 + e_n, e_2 + e_3 + e_n\}$.

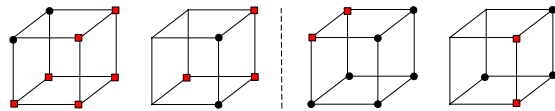
Consider the straight circuit

$$K_1 := (e_2 : 1, 3, 4, 5, \dots, n - 1, 2).$$

By (2), K_1 is infeasible and $K_1 \Delta e_n$ is feasible. The induction hypothesis applied to K_1 implies that the feasible points in $\{x : x_5 = \dots = x_n = 0, x_2 = 1\}$ form a hypercube, implying in turn that $\{e_2 + e_4, e_1 + e_2 + e_4\} \subseteq \bar{S}$, and so by (2), $\{e_2 + e_4 + e_n, e_1 + e_2 + e_4 + e_n\} \subseteq S$:



We claim that $e_1 + e_3 + e_4 \in \bar{S}$. Suppose for a contradiction that $e_1 + e_3 + e_4 \in S$. By (2), $e_1 + e_3 + e_4 + e_n \in \bar{S}$. By the Sight Propagation Lemma, $e_1 + e_4 \in S$, and so by (2), $e_1 + e_4 + e_n \in \bar{S}$:



Consider the 3-dimensional restriction of S containing $e_3 + e_1, e_3 + e_4, e_3 + e_n$; as this restriction is neither P_3 nor $R_{1,1}$, it follows that $e_3 + e_4 \in S$. If $e_4 \in S$, then as S does not have antipodal points and $\mathbf{0}, e_1, e_1 + e_3$ satisfy (\diamond) by (2), the 3-dimensional restriction of S containing $\{e_1, e_3, e_4\} \triangle \mathbf{1}$ is fragile, thereby contradicting Theorem 1.8 (ii). Otherwise, $e_4 \notin \bar{S}$. By the Sight Propagation Lemma, $e_2 + e_3 + e_4 \in S$. Consider the straight circuit

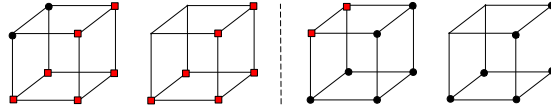
$$K_2 := (\mathbf{0} : 4, 2, 1, 3, 5, \dots, n-1).$$

By (2), K_2 is infeasible and $K_2 \triangle e_n$ is feasible. However, the 3-dimensional restriction of S containing $e_4 + e_1, e_4 + e_2, e_4 + e_3$ is a D_3 whose infeasible points all belong to K_3 , so by Proposition 5.3, S has an $R_{1,1}, R_5$ restriction, a contradiction. Thus, $e_1 + e_3 + e_4 \in \bar{S}$, and so by (2), $e_1 + e_3 + e_4 + e_n \in S$.

Consider the straight circuit

$$K_3 := (\mathbf{0} : 1, 3, 4, 2, 5, \dots, n-1).$$

By (2), K_3 is infeasible and $K_3 \triangle e_n$ is feasible. The induction hypothesis applied to K_3 tells us that the feasible points in $\{x : x_5 = \dots = x_n = 0, x_2 = 0\}$ form a hypercube, implying in turn that $\{e_4, e_1 + e_4\} \subseteq \bar{S}$. By (2), $\{e_4 + e_n, e_1 + e_4 + e_n\} \subseteq S$:



Resistance now implies that the feasible points in $\{x : x_5 = \dots = x_n = 0\}$ form a hypercube, thereby completing the induction step.

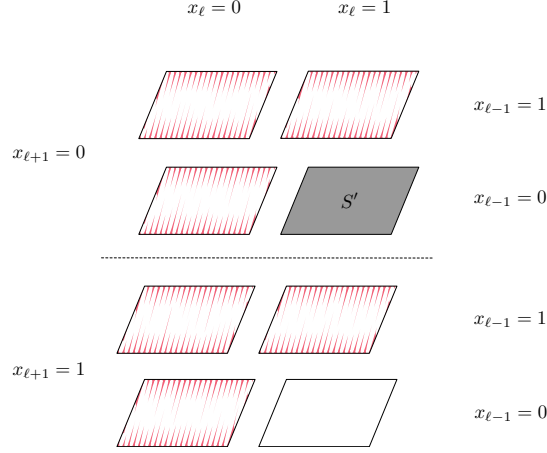
(v) After possibly relabeling coordinates $1, \dots, \ell$ and rerouting K according to (1), we may assume that $S' = \{x : x_{\ell-1} = 0, x_\ell = 1, x_{\ell+2} = \dots = x_n = 0\}$ while K remains as $(\mathbf{0} : 1, \dots, n-1)$. As $\ell - 2 \geq 2$, the points in S' are active in directions 1, 2. Let us apply the induction hypothesis to the straight infeasible circuit K but with a different starting point $(e_1 : 2, \dots, n-1, 1)$, given that the points in $\{x : x_{\ell+1} = \dots = x_n = 0, x_1 = 1\}$ satisfy (\diamond) and its feasible points $S' \cap \{x : x_1 = 1\}$ form a hypercube; and to the straight infeasible circuit $K_4 := (e_2 : 1, 3, \dots, n-1, 2)$ satisfying $V(K_4 \triangle e_n) \subseteq S$, given that the points in $\{x : x_{\ell+1} = \dots = x_n = 0, x_2 = 1\}$ satisfy (\diamond) and its feasible points $S' \cap \{x : x_2 = 1\}$ form a hypercube. The induction hypothesis implies that

(5) the points in $\{x : x_{\ell+2} = \dots = x_n = 0, x_1 = 1\}$ satisfy (\diamond) and its feasible points form a hypercube,

and that the points in $\{x : x_{\ell+2} = \dots = x_n = 0, x_2 = 1\}$ satisfy (\diamond) and its feasible points form a hypercube. The latter implies in particular that $\sum_{i=1}^{\ell+1} e_i - e_1 \in \bar{S}$. We will next apply the induction hypothesis to the straight infeasible circuit $K_5 := (\mathbf{0} : 2, \dots, \ell+1, 1, \ell+1, \dots, n-1)$ satisfying $V(K_5 \triangle e_n) \subseteq S$, given that the points in $\{x : x_{\ell+1} = \dots = x_n = 0, x_1 = 0\}$ satisfy (\diamond) and its feasible points $S' \cap \{x : x_1 = 0\}$ form a hypercube. The induction hypothesis tells us that

(6) the points in $\{x : x_{\ell+2} = \dots = x_n = 0, x_1 = 0\}$ satisfy (\diamond) and its feasible points form a hypercube.

By (5) and (6), the points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ satisfy (\diamond) and the feasible points of $\{x : x_{\ell+2} = \dots = x_n = 0\}$ are contained in $S' \cup (S' \triangle e_{\ell+1})$:



After applying the Plane Propagation Lemma to the 0-restriction of S over coordinates $\ell + 2, \dots, n$, we see that the feasible points in $\{x : x_{\ell+2} = \dots = x_n = 0\}$ must in fact form a hypercube, thereby completing the induction step. This finishes the proof of the lemma. \square

We are now ready to prove the main result of this section:

Theorem 5.5. *Take an integer $n \geq 4$ and a resistant set $S \subseteq \{0, 1\}^n$ that is nonpolar. Assume that there is a straight infeasible circuit K of length $2(n - 1)$ contained in $\{x : x_n = 0\}$ such that $V(K \triangle e_n) \subseteq S$. Then S has one of $\{R_{k,1} : 1 \leq k \leq n - 2\} \cup \{R_5\}$ as a restriction.*

Proof. After a possible twisting and relabeling, we may assume that $K = (\mathbf{0} : 1, 2, \dots, n - 1)$.

Claim. *If $n = 4$, then $S \cong R_{2,1}$.*

Proof of Claim. Suppose that $n = 4$. As $V(K) \subseteq \bar{S}$ and $V(K \triangle e_4) \subseteq S$, it follows that

$$\{0000, 1000, 1100, 1110, 0110, 0010\} \subseteq \bar{S} \quad \text{and} \quad \{0001, 1001, 1101, 1111, 0111, 0011\} \subseteq S.$$

Since S is nonpolar, $|\{1010, 0100\} \cap S| \geq 1$. Since 1010, 0100 are both adjacent to a vertex of K , it follows from the Straight Circuit Lemma that $\{1010, 0100\} \subseteq S$ and $\{0101, 1011\} \subseteq \bar{S}$, implying in turn that $S \cong R_{2,1}$, as required. \diamond

We may therefore assume that $n \geq 5$. By the Straight Circuit Lemma, the points of $\{x : x_3 = \dots = x_n = 0\}$ satisfy (\diamond) . Also, as $\{x : x_3 = \dots = x_n = 0\}$ contains at most one feasible point, the hypotheses of Lemma 5.4 hold for $\ell = 2$. If S has one of $\{R_{k,1} : 1 \leq k \leq n - 3\} \cup \{R_5\}$ as a restriction, then we are done. Otherwise, after applying Lemma 5.4 for $\ell = 2, \dots, n - 3$ in this order, we see that the points in $\{x : x_{n-1} = x_n = 0\}$

satisfy (\diamond) , implying in turn that all the points in $\{0, 1\}^n$ satisfy (\diamond) , and that $S' := S \cap \{x : x_{n-1} = x_n = 0\}$ is a hypercube. Since S is nonpolar and (\diamond) holds, it follows that $S' \neq \emptyset$. Pick a closest pair of feasible points a, b such that $a \in S'$ and $b \in (S \cap \{x : x_n = 0\}) - S' = S' \Delta \mathbf{1} \Delta e_n$. Notice that $\text{dist}(a, b) \geq 2$. It follows from (\diamond) that the restriction of S containing $a, b \Delta e_n$ as antipodal points is one of $\{R_{k,1} : 1 \leq k \leq n-2\}$. In either one of the two cases, S has one of $\{R_{k,1} : 1 \leq k \leq n-2\} \cup \{R_5\}$ as a restriction, as required. \square

6 Proofs of Theorems 1.16 and 1.17

Let us start with the following result:

Proposition 6.1. *Take an integer $n \geq 3$ and a resistant set $S \subseteq \{0, 1\}^n$ without antipodal points. If every straight infeasible path has length at most $n-2$, then S has an $R_{1,1}$ restriction.*

Proof. Let $m \leq n-2$ be the maximum length of a straight infeasible path. Since S does not have antipodal points, it follows that

(\star) every straight feasible path has length at most m .

Let $P := (v_0, v_1, \dots, v_m)$ be a maximum length straight infeasible path. After a possible twisting and relabeling, we may assume that $v_0 = \mathbf{0}$ and $v_j = v_{j-1} \Delta e_j$ for $j \in [m]$. Our maximal choice of P implies that for each $j \in \{m+1, \dots, n\}$, the points $v_0 \Delta e_j, v_m \Delta e_j$ are feasible. Thus, by the Path Propagation Lemma,

(\diamond) for each $j \in \{m+1, \dots, n\}$, $P \Delta e_j$ is a feasible path.

If $m = n-2$, then $v_0 \Delta e_{m+1} = e_{n-1}$ and $v_m \Delta e_{m+2} = \mathbf{1} - e_{n-1}$ are feasible points by (\diamond) , which cannot be the case as there are no antipodal feasible points. Thus, $m \leq n-3$. Let

$$R := S \cap \{x : x_i = 0, i \notin \{m+1, m+2, m+3\}\}.$$

By assumption, $\mathbf{0} \notin R$, and by (\diamond) , $e_{m+1}, e_{m+2}, e_{m+3} \in R$. Moreover, by (\star) , $P \Delta e_{m+1}, P \Delta e_{m+2}, P \Delta e_{m+3}$ are maximal straight feasible paths, so $e_{m+1} \Delta e_{m+2}, e_{m+2} \Delta e_{m+3}, e_{m+3} \Delta e_{m+1} \notin R$. As S is resistant, it does not have a fragile restriction by Theorem 1.8 (ii), so $e_{m+1} \Delta e_{m+2} \Delta e_{m+3} \in R$. As a result, after dropping coordinates $[n] - \{m+1, m+2, m+3\}$ from R we obtain an $R_{1,1}$, so S has an $R_{1,1}$ restriction, as required. \square

Using Theorem 5.5 and Proposition 6.1, we prove the following:

Theorem 6.2. *Take an integer $n \geq 3$ and a resistant set $S \subseteq \{0, 1\}^n$ that is nonpolar. If every straight infeasible path has length at most $n-1$, then S has one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction.*

Proof. If there is no straight infeasible path of length $n-1$, then S has an $R_{1,1}$ restriction by Proposition 6.1, so we are done. Otherwise, there is a straight infeasible path $P := (v_0, v_1, \dots, v_{n-1})$ of length $n-1$, which by assumption is maximal. After a possible relabeling and twisting, if necessary, we may assume that $V(P) \subseteq \{x : x_n = 0\}$. Maximality of P implies that $v_0 \Delta e_n, v_{n-1} \Delta e_n$ are feasible, so by the Path Propagation Lemma,

$P\Delta e_n$ is a feasible path. As S does not contain antipodal points, it follows that the path $Q := P\Delta e_n\Delta \mathbf{1}$ is infeasible. Since Q is a straight infeasible (v_{n-1}, v_0) -path, and $v_{n-1}\Delta e_n, v_0\Delta e_n \in S$, we get from the Path Propagation Lemma that $Q\Delta e_n$ is a feasible path. Consider the straight infeasible circuit $K := P \cup Q$ of length $2(n-1)$ contained in $\{x : x_n = 0\}$. We just showed that $V(K\Delta e_n) \subseteq S$. Thus, by Theorem 5.5, S has one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction, as required. \square

We are now ready to prove Theorem 1.16, stating that up to isomorphism, $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are the only half-dense strictly nonpolar sets that are resistant:

Proof of Theorem 1.16. Take an integer $n \geq 3$ and a half-dense strictly nonpolar set $S \subseteq \{0, 1\}^n$ that is resistant. Since S is nonpolar and half-dense, it follows that for each $x \in \{0, 1\}^n$, one of $x, \mathbf{1} - x$ is feasible and the other is infeasible. In particular, there is no antipodal pair of infeasible points. Since a straight path of length n has antipodal points as ends, it therefore follows that every straight infeasible path has length at most $n - 1$. Hence, as S is resistant and nonpolar, Theorem 6.2 implies that S has one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction. As $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are nonpolar, and S is strictly nonpolar, S must be isomorphic to one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$, as required. \square

Next we prove Theorem 1.17, which states the following:

Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0, 1\}^{n_1}, S_2 \subseteq \{0, 1\}^{n_2}$, where $S_1, \overline{S_1}, S_2, \overline{S_2}$ are nonempty and resistant. Then $S_1 * S_2$ is strictly polar if, and only if, $S_1 * S_2$ has none of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction.

This theorem is by and large a consequence of Corollary 4.15 and Theorem 1.16. The proof also relies on the following result:

Proposition 6.3 ([1], Proposition 5.11). *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0, 1\}^{n_1}, S_2 \subseteq \{0, 1\}^{n_2}$, where $S_1, \overline{S_1}, S_2, \overline{S_2}$ are nonempty. If one of $S_1, \overline{S_1}, S_2, \overline{S_2}$ is not strictly connected, then $S_1 * S_2$ has one of $\{R_{k,1} : k \geq 1\}$ as a restriction.*

We will also need the following result:

Theorem 6.4 ([1], Theorem 1.18 (2)). *Take integers $n_1, n_2 \geq 1$ and sets $S_1 \subseteq \{0, 1\}^{n_1}, S_2 \subseteq \{0, 1\}^{n_2}$, where $S_1 * S_2$ is strictly nonpolar. Then either $n_1 = 1$ or $n_2 = 1$. In particular, $S_1 * S_2$ is half-dense.*

We are now ready to prove the final result of the paper, Theorem 1.17:

Proof of Theorem 1.17. (\Rightarrow) holds trivially. (\Leftarrow) Assume that $S_1 * S_2$ has a nonpolar restriction. We need to show that $S_1 * S_2$ has one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ as a restriction. If one of $S_1, \overline{S_1}, S_2, \overline{S_2}$ is not strictly connected, then by Proposition 6.3, $S_1 * S_2$ has one of $\{R_{k,1} : k \geq 1\}$ as a restriction, so we are done. Otherwise, $S_1, \overline{S_1}, S_2, \overline{S_2}$ are strictly connected. Since they are also resistant, Corollary 4.15 implies that $S_1, \overline{S_1}, S_2, \overline{S_2}$ are strictly polar.

For each $i \in \{1, 2\}$, take an integer $m_i \geq 0$ and a restriction $R_i \subseteq \{0, 1\}^{m_i}$ so that $R_1 * R_2$ is a strictly nonpolar restriction of $S_1 * S_2$. As restrictions of S_1 and S_2 , R_1 and R_2 are both polar, implying in turn that $m_1 \geq 1$ and $m_2 \geq 1$. Therefore, $R_1 * R_2$ is half-dense by Theorem 6.4. As restrictions of $S_1, \overline{S_1}, S_2, \overline{S_2}$, the sets $R_1, \overline{R_1}, R_2, \overline{R_2}$ are resistant by Remark 3.2. Thus, $R_1 * R_2$ is resistant by Theorem 1.7 (3). As a result, $R_1 * R_2$ is isomorphic to one of $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ by Theorem 1.16, in turn finishing the proof. \square

7 Open questions and concluding remarks

We showed in Theorem 1.16 that $\{R_{k,1} : k \geq 1\} \cup \{R_5\}$ are, up to isomorphism, the only strictly nonpolar resistant sets that are half-dense. Question 1.15, asking for all of the resistant strictly nonpolar sets, remains open. In fact, we cannot even answer the following question:

Question 7.1. *Is there a nonpolar resistant set $S \subseteq \{0, 1\}^n$ such that $|S| < 2^{n-1}$?*

Let \mathcal{C} be a clutter over ground set E . We say that \mathcal{C} has the *max-flow min-cut property* if the set covering linear program

$$\begin{array}{ll} \min & w^\top x \\ \text{s.t.} & x(C) \geq 1 \quad C \in \mathcal{C} \\ & x \geq \mathbf{0} \end{array}$$

is totally dual integral. Since total dual integrality implies primal integrality [20, 15], the max-flow min-cut property implies idealness ([9], Theorem 4.1). In fact, it can be readily checked that the max-flow min-cut property also implies the packing property [8]. The Replication Conjecture of Conforti and Cornuéjols [8] predicts the converse should hold, that

(?) the packing property implies the max-flow min-cut property. (?)

Corollary 1.12 showed that if a set is resistant and strictly polar, then its cuboid has the packing property. So the Replication Conjecture, if true, would imply the following:

Conjecture 7.2. *If a set is resistant and strictly polar, then its cuboid has the max-flow min-cut property.*

It seems to us that to answer Questions 1.15, 7.1 and Conjecture 7.2, we need to have a structure theorem for resistant sets. In a sequel paper, we provide structure theorems for natural classes of resistant sets [2].

Many of the theorems proved in this paper stemmed from propagations running in resistant sets. Do cube-ideal sets in general have propagation features? The answer is yes; a weaker form of the Sight Propagation Lemma holds for cube-ideal sets. To elaborate, take an integer $n \geq 3$. A *delta of dimension n* is the clutter over ground set $[n]$ whose members are

$$\Delta_n := \{\{1, 2\}, \{1, 3\}, \dots, \{1, n\}, \{2, 3, \dots, n\}\}.$$

Δ_n is a non-ideal clutter as $\left(\frac{n-2}{n-1} \frac{1}{n-1} \cdots \frac{1}{n-1}\right)$ is a fractional extreme point of the corresponding set covering polyhedron. As a result, ideal clutters do not have a delta minor. We will need the following tool for finding delta minors:

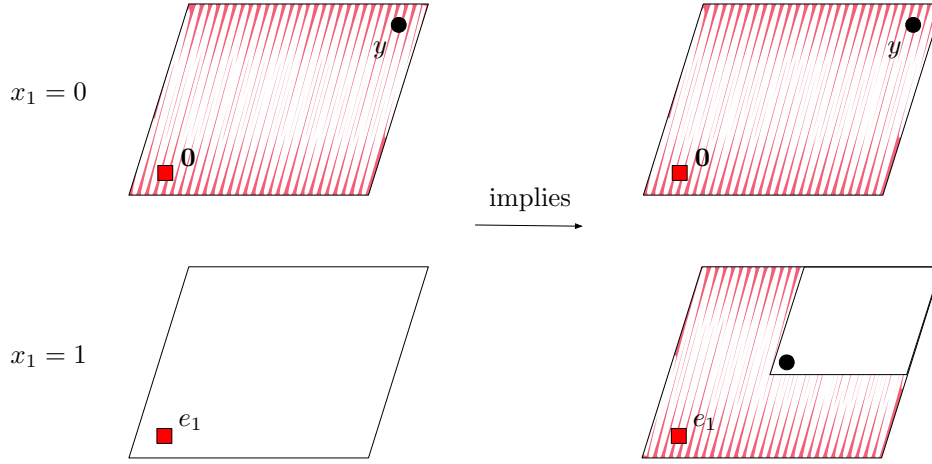


Figure 6: An illustration of Corollary 7.4, where $\{z \in S : z \leq y + e_1, z_1 = 1\}$ has a unique minimal point.

Theorem 7.3 ([3], Theorem 2.1). *Let \mathcal{C} be a clutter. If there are distinct members C_1, C_2, C and an element e such that $e \in C_1 \cap C_2$, $e \notin C$ and $C_1 \cup C_2 \subseteq \{e\} \cup C$, then \mathcal{C} has a delta minor through e .*

As a consequence, we get the following weakening of Remark 4.7 for cube-ideal sets, which is illustrated in Figure 6.

Corollary 7.4. *Take an integer $n \geq 1$ and a cube-ideal set $S \subseteq \{0, 1\}^n$, where $\mathbf{0}, e_1$ are infeasible. Assume that y is a minimal feasible point such that $y_1 = 0$. Then $\{z \in S : z \leq y + e_1, z_1 = 1\}$ has at most one minimal point.*

Proof. Observe that every minimal point of $\{z \in S : z \leq y + e_1, z_1 = 1\}$, other than $y + e_1$, is also a minimal point of S . Suppose for a contradiction that z^1, z^2 are distinct minimal points of $\{z \in S : z \leq y + e_1, z_1 = 1\}$. Then z^1, z^2 must be different from $y + e_1$, so they are minimal points of S . Pick members $C, C_1, C_2 \in \text{ind}(S)$ such that $y = \chi_C, z^1 = \chi_{C_1}, z^2 = \chi_{C_2}$. Notice that $1 \in C_1 \cap C_2, 1 \notin C$ and $C_1 \cup C_2 \subseteq \{1\} \cup C$. Theorem 7.3 implies that $\text{ind}(S)$ has a delta minor, so $\text{ind}(S)$ is non-ideal. Theorem 1.2 now applies and tells us that S is not cube-ideal, a contradiction. \square

Acknowledgements

This work was supported in parts by ONR grant 00014-15-12082, NSF grant CMMI-1560828, and an NSERC CGS D3 grant.

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