Deltas, delta minors and delta free clutters

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Abstract

For an integer $n \geq 3$, the clutter $\Delta_n := \big\{\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}\big\}$ is called a *delta of dimension* n, whose members are the lines of a degenerate projective plane. In his seminal paper on non-ideal clutters, Alfred Lehman manifested the role of the deltas as a distinct class of minimally non-ideal clutters [DIMACS, 1990]. A clutter is *delta free* if it has no delta minor. Binary clutters, ideal clutters and clutters with the packing property are examples of delta free clutters. In this paper, after introducing and studying basic geometric notions defined on clutters, we will investigate the surprising geometric attributes of the deltas, delta minors and delta free clutters. We will also state some conjectures on identically self-blocking clutters.

1 Introduction

Let E be a finite set of *elements*, and $\mathcal{C}$ a family of subsets of E called *members*. If no member is contained in another one, Edmonds and Fulkerson call $\mathcal{C}$ a *clutter* over *ground set* $E(\mathcal{C}) := E$ [12]. The *incidence matrix* of $\mathcal{C}$, denoted by $M(\mathcal{C})$, is the $0,1$ matrix whose columns are labeled by the elements and whose rows are the incidence vectors of the members. A *cover* of the clutter $\mathcal{C}$ is a subset of E that intersects every member. The family of all (inclusion-wise) minimal covers of $\mathcal{C}$ is another clutter over the same ground set, called the *blocker of* $\mathcal{C}$ and denoted by $b(\mathcal{C})$. It is well-known that the blocking relation is an involution, that is, $b(b(\mathcal{C})) = \mathcal{C}$ [14, 12]. Given disjoint subsets $I, J \subseteq E$, we refer to the clutter

$$\mathcal{C} \setminus I/J := \text{ the minimal sets of } \big\{C - J : C \in \mathcal{C}, C \cap I = \emptyset\big\}^1$$

as the *minor* of $\mathcal{C}$ obtained after *deleting* I and *contracting* J. If $I \cup J \neq \emptyset$, then the minor is *proper*. It can be readily checked that $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$ [20].

A clutter $\mathcal{C}$ is *binary* if, for all members C_1, C_2, C_3, the symmetric difference $C_1 \triangle C_2 \triangle C_3$ contains a member.[2] It can be readily checked that if a clutter is binary, then so is every minor of it [15, 20]. It is known that a clutter $\mathcal{C}$ is binary if, and only if, $|C \cap B|$ is odd for all $C \in \mathcal{C}$ and $B \in b(\mathcal{C})$ [15]. As a result, a clutter is binary if and only if its blocker is binary. For instance, given a graph G and distinct vertices s and t, the family of st-paths

[1] $A - B := \{a \in A : a \notin B\}$

[2] $A_1 \triangle \cdots \triangle A_k$ is the set of elements that belong to an odd number of the A_i's.

is a clutter over ground set $E(G)$. It can be readily checked that the clutter of st-cuts is the blocker. Since every st-path and every st-cut have an odd number of edges in common, it follows that the clutter of st-paths is binary.

The set

$$Q(\mathcal{C}) := \left\{x \in \mathbb{R}^E_+ : M(\mathcal{C})x \geq \mathbf{1}\right\} = \left\{x \in \mathbb{R}^E_+ : \sum \left(x_e : e \in C\right) \geq 1 \quad C \in \mathcal{C}\right\}$$

is the *set covering polyhedron* of $\mathcal{C}$. Observe that the $0,1$ points in $Q(\mathcal{C})$ are precisely the characteristic vectors of the covers of $\mathcal{C}$, and that the integral extreme points of $Q(\mathcal{C})$ are precisely the characteristic vectors of the minimal covers of $\mathcal{C}$. Notice that $Q(\mathcal{C} \setminus I/J)$ is obtained from $Q(\mathcal{C})$ after projecting away $(x_i : i \in I)$, restricting to $\{x : x_j = 0 \;\; j \in J\}$ and then dropping the coordinates in J. Coined by Cornuéjols and Novick, clutter $\mathcal{C}$ is *ideal* if $Q(\mathcal{C})$ is an integral polyhedron [10]. It follows from the Width-Length Inequality that a clutter is ideal if and only if its blocker is ideal [16] (see also [8], Theorem 1.21). Due to our polyhedral interpretation of minor operations, if a clutter is ideal then so is every minor of it. A clutter is *minimally non-ideal (mni)* if it is non-ideal but every proper minor is ideal. Note that a clutter is ideal if and only if it has no mni minor.

The *covering number* of $\mathcal{C}$, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover. The *packing number* of $\mathcal{C}$, denoted $\nu(\mathcal{C})$, records the maximum number of (pairwise) disjoint members of $\mathcal{C}$. Since a cover pick a distinct element from every member of a packing,

$$\tau(\mathcal{C}) \geq \nu(\mathcal{C}).$$

If equality holds here, we say that $\mathcal{C}$ *packs*. We say that $\mathcal{C}$ has the *packing property* if every minor of it (including $\mathcal{C}$ itself) packs [9].

Theorem 1.1 (see [8], Theorem 1.8)**.** *A clutter with the packing property is ideal.*

This result is a fascinating consequence of Alfred Lehman's result on mni clutters [17]. A clutter is *minimally non-packing (mnp)* if it does not pack but every proper minor of it packs. Notice once again that a clutter has the packing property if and only if it has no mnp minor. Since every clutter with the packing property is ideal, it follows that an mnp clutter is either ideal or mni.

1.1 Deltas and delta free clutters

Two clutters $\mathcal{C}_1, \mathcal{C}_2$ are *isomorphic*, denoted $\mathcal{C}_1 \cong \mathcal{C}_2$, if $\mathcal{C}_1$ may be obtained from $\mathcal{C}_2$ after relabeling the ground set $E(\mathcal{C}_2)$. For an integer $n \geq 3$, any clutter isomorphic to the clutter over ground set $[n]$ whose members are

$$\Delta_n := \big\{\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}\big\},$$

is referred to as a *delta* of *dimension* n.[3] The members of Δ_n correspond to the lines of a degenerate projective plane over points $[n]$. Observe that $b(\Delta_n) = \Delta_n$, that Δ_n is non-binary as the member $\{1,2\}$ intersects the minimal cover $\{1,2\}$ twice, that Δ_n is non-ideal as $\left(\frac{n-2}{n-1} \frac{1}{n-1} \cdots \frac{1}{n-1}\right) \in Q(\Delta_n)$ is an extreme point, and that Δ_n is non-packing as $2 = \tau(\Delta_n) > \nu(\Delta_n) = 1$. In his seminal characterization of ideal clutters, Alfred

[3] $[n] := \{1,\ldots,n\}$

Lehman showed how the deltas are a significant class of mni clutters, a class that behaves quite differently from all other mni clutters (see Theorem 6.3) [17].

A clutter is *delta free* if it has no delta minor. For example, binary clutters, ideal clutters and clutters with the packing property are delta free. Recently, delta free clutters have surfaced and shown relevance to some conjectures in the field. A major open question in our field is the 1993 Conforti and Cornuéjols' *Replication Conjecture*, predicting that replication preserves the packing property [6].[4] Properties of delta free clutters have led to a result similar to (and implied by) the Replication Conjecture, stating that splitting preserves the packing property ([2], Theorem 4.7).[5] In an attempt to resolve the Replication Conjecture, in 2000, Cornuéjols, Guenin and Margot made a stronger conjecture that an ideal mnp should always have covering number two [9]. Delta free clutters have exposed among ideal mnp clutters the significance of those with covering number two, and revealed a characterization of ideal mnp clutters of covering number two [1].

All these connections, including Lehman's manifestation of deltas as a distinct class of mni clutters, beg to the following thematic questions: what can be said about delta free clutters? and is there a structure theorem for such clutters? Answering these questions may not be so out of reach as testing delta-free-ness belongs to P [1]. This is in contrast with the surprising result of Ding, Feng and Zang from 2008, that testing idealness is a co-NP-complete problem [11]. Perhaps studying delta free clutters provides the tools needed to tackle some of the major conjectures in the field.

We continue the study of deltas, delta minors, as well as delta free clutters. We will introduce and study some very basic geometric notions defined on clutters. Along the way, we see an intimate link between the geometry of clutters and the presence of delta minors. We will also explore the geometry of delta free clutters, and of specific classes such as binary clutters, ideal clutters and clutters with the packing property.

1.2 Lifts, projections, entanglement, common projections and common lifts

Here we introduce some geometric notions defined on clutters.

Lifts and projections. Take a clutter $\mathcal{C}$ and an element $e \in E(\mathcal{C})$. Let $\mathcal{C}_1 := \mathcal{C} \setminus e$ and $\mathcal{C}_2 := \mathcal{C}/e$. We say that $\mathcal{C}_1$ is a *lift* of $\mathcal{C}_2$, and $\mathcal{C}_2$ is a *projection* of $\mathcal{C}_1$. That is, a lift is what is obtained after coextending by an element and then deleting the element, while a projection is what is obtained after extending by an element and then contracting the element.

Proposition 1.2. *Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set E. The following statements are equivalent:*

(i) *$\mathcal{A}$ is a lift of $\mathcal{B}$,*

(ii) *$b(\mathcal{A})$ is a projection of $b(\mathcal{B})$,*

[4] To *replicate* an element e of $\mathcal{C}$ is to introduce a new element $\bar{e}$ and replace $\mathcal{C}$ by the clutter $\mathcal{C} \cup \{C \triangle \{e, \bar{e}\} : e \in C \in \mathcal{C}\}$.

[5] Take a clutter $\mathcal{C}$ and an element e. Partition the members of $\mathcal{C}$ containing e into parts $\mathcal{C}_1, \mathcal{C}_2$ such that for all $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$, $(C_1 \cup C_2) - \{e\}$ contains another member. To *split* e is to introduce a new element $\bar{e}$ and replace $\mathcal{C}$ by $\{C \in \mathcal{C} : C \notin \mathcal{C}_2\} \cup \{C \triangle \{e, \bar{e}\} : C \in \mathcal{C}_2\}$.

(iii) every member of $\mathcal{A}$ contains a member of $\mathcal{B}$,

(iv) $Q(\mathcal{A})$ contains $Q(\mathcal{B})$.

Proof. Let $\mathcal{C}$ be a clutter and take an element $e \in E(\mathcal{C})$. **(i)** $\Leftrightarrow$ **(ii)** follows immediately from the following observation: $\mathcal{A} = \mathcal{C} \setminus e$ and $\mathcal{B} = \mathcal{C}/e$ if and only if $b(\mathcal{A}) = b(\mathcal{C})/e$ and $b(\mathcal{B}) = b(\mathcal{C}) \setminus e$. **(i)** $\Leftrightarrow$ **(iii)**: Every member of $\mathcal{C} \setminus e$ contains a member of $\mathcal{C}/e$, so (i) implies (iii). Conversely, assume that (iii) holds. Take a new element label $f \notin E$, and let $\mathcal{C}'$ be the clutter over ground set $E \cup \{f\}$ whose members are the minimal sets of $\big\{\{f\} \cup B : B \in \mathcal{B}\big\} \cup \big\{A : A \in \mathcal{A}\big\}$. Then $\mathcal{C}' \setminus f = \mathcal{A}$, and as (iii) holds we have $\mathcal{C}'/f = \mathcal{B}$, so (i) holds.

(iii) $\Rightarrow$ **(iv)**: Assume that (iii) holds. Then every defining inequality of $Q(\mathcal{A})$ is valid for every point in $Q(\mathcal{B})$: if $x \in Q(\mathcal{B})$ and $A \in \mathcal{A}$, then A contains a member $B \in \mathcal{B}$, so $x(A) \geq x(B) \geq 1$. Thus $Q(\mathcal{B}) \subseteq Q(\mathcal{A})$, so (iv) holds. **(iv)** $\Rightarrow$ **(i)**: Assume that (iv) holds. Then every minimal cover of $\mathcal{B}$ is a cover of $\mathcal{A}$, that is, every member of $b(\mathcal{B})$ contains a member of $b(\mathcal{A})$. As (i) and (iii) are equivalent, it follows that $b(\mathcal{B})$ is a lift of $b(\mathcal{A})$, and therefore as (i) and (ii) are equivalent, we get that $\mathcal{A}$ is a lift of $\mathcal{B}$, so (i) holds. $\square$

Entanglement. Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set. We say that $\mathcal{A}$ *entangles* $\mathcal{B}$ if, for all $A \in \mathcal{A}$ and $A' \in b(\mathcal{A})$ such that $|A \cap A'| = 1$, either A contains a member of $\mathcal{B}$, or A' is a cover of $\mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ entangle one another, then they are *tangled*. Notice that, if $\mathcal{A}$ entangles $\mathcal{B}$ then $b(\mathcal{A})$ entangles $b(\mathcal{B})$, and as a result, if $\mathcal{A}, \mathcal{B}$ are tangled then so are $b(\mathcal{A}), b(\mathcal{B})$.

Remark 1.3. *If $\mathcal{A}$ is a lift of $\mathcal{B}$, then $\mathcal{A}, \mathcal{B}$ are tangled.*

Proof. Assume that $\mathcal{A}$ is a lift of $\mathcal{B}$. By Proposition 1.2 (iii) $\mathcal{A}$ entangles $\mathcal{B}$. By (ii) $b(\mathcal{B})$ is a lift of $b(\mathcal{A})$, so by (iii) $b(\mathcal{B})$ entangles $b(\mathcal{A})$, implying in turn that $\mathcal{B}$ entangles $\mathcal{A}$. Thus $\mathcal{A}$ and $\mathcal{B}$ are tangled. $\square$

We view this as the trivial instance of tangled clutters. In §5 we will see that if a binary clutter entangles another, then the entanglement must be trivial. There are of course non-trivial instances of tangled clutters:

Proposition 1.4. *Let $\mathcal{C}$ be a clutter over ground set E. Suppose $\mathcal{C}$ has no cover of cardinality one, and there are covers A, B that partition E. Let $\mathcal{A}$ be the clutter of the minimal sets of $\mathcal{C} \cup \{A\}$, and let $\mathcal{B}$ be the clutter of the minimal sets of $\mathcal{C} \cup \{B\}$. Then $\mathcal{A}, \mathcal{B}$ are tangled clutters, neither of which is a lift of the other.*

Proof. Since A is a cover of $\mathcal{C}$, $E - A = B$ does not contain a member of $\mathcal{C}$, implying that $B \in \mathcal{B}$ and by Proposition 1.2 (iii) that $\mathcal{B}$ is not a lift of $\mathcal{A}$. Similarly, $A \in \mathcal{A}$ and $\mathcal{A}$ is not a lift of $\mathcal{B}$. We will show that $\mathcal{A}$ entangles $\mathcal{B}$. If $C \in \mathcal{A} - \{A\}$, then either $B \subseteq C$ or $C \in \mathcal{B}$, so either way, C contains a member of $\mathcal{B}$. Now consider the member A of $\mathcal{A}$, and let A' be a minimal cover of $\mathcal{A}$ such that $|A \cap A'| = 1$. Then A' is also a cover of $\mathcal{C}$, so $|A'| > 1$. Thus $A' \cap B = A' \cap (E - A) \neq \emptyset$, implying in turn that A' is also a cover of $\mathcal{B}$. Hence, $\mathcal{A}$ entangles $\mathcal{B}$. Similarly, $\mathcal{B}$ entangles $\mathcal{A}$, so $\mathcal{A}$ and $\mathcal{B}$ are tangled. $\square$

In §2 we study the geometry enforced by entanglement and see a link between entanglement and set covering polyhedra having a convex union. Not only is entanglement closed under taking blockers, it is also a minor-closed property:

Remark 1.5. *Let $\mathcal{A},\mathcal{B}$ be clutters over the same ground set E, and take disjoint sets $I,J\subseteq E$. If $\mathcal{A}$ entangles $\mathcal{B}$, then $\mathcal{A}\setminus I/J$ entangles $\mathcal{B}\setminus I/J$.*

Proof. Suppose $\mathcal{A}$ entangles $\mathcal{B}$. Assume that $|C\cap C'|=1$ for some $C\in\mathcal{A}\setminus I/J$ and $C'\in b(\mathcal{A}\setminus I/J)$. Choose $A\in\mathcal{A}$ and $A'\in b(\mathcal{A})$ such that $C\subseteq A\subseteq C\cup J$ and $C'\subseteq A'\subseteq C'\cup I$. Then $A\cap A'=C\cap C'$ and so $|A\cap A'|=1$. As $\mathcal{A}$ entangles $\mathcal{B}$, either A contains a member of $\mathcal{B}$, or A' is a cover of $\mathcal{B}$, implying in turn that either C contains a member of $\mathcal{B}\setminus I/J$, or C' is a cover of $\mathcal{B}\setminus I/J$. Since this is true for all such C and C', it follows that $\mathcal{A}\setminus I/J$ entangles $\mathcal{B}\setminus I/J$. $\square$

In §3 and §4 we provide an excluded-minor characterization of entanglement. Along the way, we will see a link between entanglement and delta minors.

Greatest common projections and least common lifts. Let $\mathcal{A},\mathcal{B}$ be clutters over the same ground set. The *greatest common projection of $\mathcal{A}$ and $\mathcal{B}$*, denoted $\mathrm{GCP}(\mathcal{A},\mathcal{B})$, is the clutter over the same ground set whose members are the minimal sets in $\mathcal{A}\cup\mathcal{B}$. The *least common lift of $\mathcal{A}$ and $\mathcal{B}$*, denoted $\mathrm{LCL}(\mathcal{A},\mathcal{B})$, is the clutter over the same ground set whose members are the minimal sets in $\{A\cup B: A\in\mathcal{A},B\in\mathcal{B}\}$. Our choice of terminology is justified in the corollary below. For example, given clutters $\big\{\{1\},\{2,3\}\big\}$ and $\big\{\{1,2\},\{3\}\big\}$, their greatest common projection is $\big\{\{1\},\{3\}\big\}$ while their least common lift is $\big\{\{1,2\},\{2,3\},\{3,1\}\big\}$. For the extreme cases, notice that $\mathrm{GCP}(\{\emptyset\},\mathcal{B})=\{\emptyset\}$ and $\mathrm{LCL}(\{\emptyset\},\mathcal{B})=\mathcal{B}$, while for the blocker $\{\}$ of $\{\emptyset\}$, $\mathrm{GCP}(\{\},\mathcal{B})=\mathcal{B}$ and $\mathrm{LCL}(\{\},\mathcal{B})=\{\}$.

Proposition 1.6. *Let $\mathcal{A},\mathcal{B}$ be clutters over the same ground set E. Then the following statements hold:*

(1) $b(\mathrm{GCP}(\mathcal{A},\mathcal{B}))=\mathrm{LCL}(b(\mathcal{A}),b(\mathcal{B}))$ and $b(\mathrm{LCL}(\mathcal{A},\mathcal{B}))=\mathrm{GCP}(b(\mathcal{A}),b(\mathcal{B}))$,

(2) $Q(\mathcal{A})\cap Q(\mathcal{B})=Q(\mathrm{GCP}(\mathcal{A},\mathcal{B}))$,

(3) $Q(\mathcal{A})\cup Q(\mathcal{B})\subseteq Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$,

(4) for disjoint $I,J\subseteq E$,

$$\mathrm{GCP}(\mathcal{A}\setminus I/J,\mathcal{B}\setminus I/J)=\mathrm{GCP}(\mathcal{A},\mathcal{B})\setminus I/J$$

and

$$\mathrm{LCL}(\mathcal{A}\setminus I/J,\mathcal{B}\setminus I/J)=\mathrm{LCL}(\mathcal{A},\mathcal{B})\setminus I/J.$$

Proof. **(1)** By taking blockers, we may just prove that $b(\mathrm{GCP}(\mathcal{A},\mathcal{B}))=\mathrm{LCL}(b(\mathcal{A}),b(\mathcal{B}))$. It suffices to show that every member of one of $b(\mathrm{GCP}(\mathcal{A},\mathcal{B})),\mathrm{LCL}(b(\mathcal{A}),b(\mathcal{B}))$ contains a member of the other. If $C\in b(\mathrm{GCP}(\mathcal{A},\mathcal{B}))$, then C is a cover of $\mathcal{A}$ as well as a cover of $\mathcal{B}$, so C contains a member of $\mathrm{LCL}(b(\mathcal{A}),b(\mathcal{B}))$. Conversely, assume that $A'\cup B'\in\mathrm{LCL}(b(\mathcal{A}),b(\mathcal{B}))$ for some $A'\in b(\mathcal{A})$ and $B'\in b(\mathcal{B})$. Since A' intersects every member of $\mathcal{A}$, and B' intersects every member of $\mathcal{B}$, it follows that $A'\cup B'$ intersects every member of $\mathrm{GCP}(\mathcal{A},\mathcal{B})$, so $A'\cup B'$ contains a member of $b(\mathrm{GCP}(\mathcal{A},\mathcal{B}))$. Thus, $b(\mathrm{GCP}(\mathcal{A},\mathcal{B}))=\mathrm{LCL}(b(\mathcal{A}),b(\mathcal{B}))$. **(2)** and **(3)** are immediate. **(4)** By (1) it suffices to show, for each $e\in E$, that $\mathrm{GCP}(\mathcal{A}\setminus e,\mathcal{B}\setminus e)=\mathrm{GCP}(\mathcal{A},\mathcal{B})\setminus e$ and $\mathrm{GCP}(\mathcal{A}/e,\mathcal{B}/e)=\mathrm{GCP}(\mathcal{A},\mathcal{B})/e$, both of which trivially hold. $\square$

In §2 we will show that equality holds in (3) if and only if $Q(\mathcal{A}) \cup Q(\mathcal{B})$ is a convex set.

Corollary 1.7. *Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set. The following statements hold:*

(1) $\mathrm{GCP}(\mathcal{A}, \mathcal{B})$ *is a common projection of $\mathcal{A}, \mathcal{B}$; and if $\mathcal{C}$ is a common projection of $\mathcal{A}, \mathcal{B}$, then $\mathcal{C}$ is also a projection of* $\mathrm{GCP}(\mathcal{A}, \mathcal{B})$*,*

(2) $\mathrm{LCL}(\mathcal{A}, \mathcal{B})$ *is a common lift of $\mathcal{A}, \mathcal{B}$; and if $\mathcal{C}$ is a common lift of $\mathcal{A}, \mathcal{B}$, then $\mathcal{C}$ is also a lift of* $\mathrm{LCL}(\mathcal{A}, \mathcal{B})$.

Proof. **(1)** follows immediately from Proposition 1.6 (2) and Proposition 1.2 (iv). **(2)** follows from (1) after applying Proposition 1.6 (1). □

Thus we may view $\mathrm{GCP}(\mathcal{A}, \mathcal{B})$ as the greatest common projection of $\mathcal{A}$ and $\mathcal{B}$, and $\mathrm{LCL}(\mathcal{A}, \mathcal{B})$ as the least common lift of $\mathcal{A}$ and $\mathcal{B}$, thereby justifying our choice of terminology.

In general, if $\mathcal{A}, \mathcal{B}$ belong to a certain minor-closed class of clutters, we cannot necessarily guarantee that $\mathrm{GCP}(\mathcal{A}, \mathcal{B}), \mathrm{LCL}(\mathcal{A}, \mathcal{B})$ also belong to that minor-closed class. However, the situation is different if $\mathcal{A}, \mathcal{B}$ are tangled clutters. In §5.1 (resp. §6) we will show that if tangled $\mathcal{A}, \mathcal{B}$ are delta free (resp. ideal), then so are $\mathrm{GCP}(\mathcal{A}, \mathcal{B})$ and $\mathrm{LCL}(\mathcal{A}, \mathcal{B})$; the converse of these statements are also proven. In §6 we study whether or not the same statement holds for the packing property. Motivated by these facts, we classify in §7 clutters $\mathcal{C}$ for which there are tangled clutters $\mathcal{A}$ and $\mathcal{B}$, neither of which is a lift of the other, such that $\mathcal{C} = \mathrm{GCP}(\mathcal{A}, \mathcal{B})$ or $\mathcal{C} = \mathrm{LCL}(\mathcal{A}, \mathcal{B})$. By then, the reader will have noticed that clutters equal to their blockers surface twice throughout the paper; in §8 we propose three conjectures on these clutters.

2 The geometry of entanglement

Recall that for clutters $\mathcal{A}, \mathcal{B}$ over the same ground set, $\mathcal{A}$ entangles $\mathcal{B}$ if for all $A \in \mathcal{A}$ and $A' \in b(\mathcal{A})$ such that $|A \cap A'| = 1$, either A contains a member of $\mathcal{B}$, or A' is a cover of $\mathcal{B}$.

Proposition 2.1. *Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set. Then the following statements are equivalent:*

(i) $\mathcal{A}$ entangles $\mathcal{B}$,

(ii) for every integral extreme point $x \in Q(\mathcal{A})$ such that $x \notin Q(\mathcal{B})$, and for every point $y \in Q(\mathcal{B})$, we have

$$\frac{1}{2}(x + y) \in Q(\mathcal{A}),$$

(iii) for every integral extreme point $x \in Q(\mathcal{A})$ such that $x \notin Q(\mathcal{B})$, and for every point $y \in Q(\mathcal{B})$, there exists an $\epsilon \in (0, \frac{1}{2}]$ such that,

$$(1 - \epsilon)x + \epsilon y \in Q(\mathcal{A}).$$

Proof. **(i) ⇒ (ii)**: Assume that $x = \chi_{A'}$ for some $A' \in b(\mathcal{A})$. Since $x \notin Q(\mathcal{B})$, A' is not a cover of $\mathcal{B}$. We need to show for each $A \in \mathcal{A}$ that $x(A) + y(A) \geq 2$. If $|A \cap A'| \geq 2$, then $x(A) + y(A) \geq 2 + 0 = 2$.

Otherwise, $|A \cap A'| = 1$. Since $\mathcal{A}$ entangles $\mathcal{B}$, A must contain a member of $\mathcal{B}$, so $y(A) \geq 1$, implying in turn that $x(A) + y(A) \geq 1 + 1 = 2$, as required. **(ii)** $\Rightarrow$ **(iii)** follows immediately by setting $\epsilon = \frac{1}{2}$. **(iii)** $\Rightarrow$ **(i)**: Assume that $|A \cap A'| = 1$ for some $A \in \mathcal{A}$ and $A' \in b(\mathcal{A})$. We need to show that either A contains a member of $\mathcal{B}$ or A' is a cover of $\mathcal{B}$. Consider the integral extreme point $x := \chi_{A'}$ of $Q(\mathcal{A})$. If $x \in Q(\mathcal{B})$, then A' is a cover of $\mathcal{B}$. Otherwise, $x \notin Q(\mathcal{B})$. It then follows from (iii) that, for each $y \in Q(\mathcal{B})$, there is an $\epsilon \in (0, \frac{1}{2}]$ such that,

$$y(A) \geq \frac{1}{\epsilon} - \frac{1-\epsilon}{\epsilon} \cdot x(A) = \frac{1}{\epsilon} - \frac{1-\epsilon}{\epsilon} \cdot |A \cap A'| = 1.$$

In particular, for each $B' \in b(\mathcal{B})$, we have $|B' \cap A| \geq 1$. So A is a cover of $b(\mathcal{B})$, implying in turn that A contains a member of $\mathcal{B}$, as required. $\square$

For points $x, y \in \mathbb{R}^n$, denote by (x, y) the open line segment joining x and y. The preceding result implies the following sufficient condition for entanglement:

Theorem 2.2. *Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set. If $Q(\mathcal{A}) \cup Q(\mathcal{B})$ is convex, then $\mathcal{A}, \mathcal{B}$ are tangled.*

Proof. Assume that $Q(\mathcal{A}) \cup Q(\mathcal{B})$ is convex. We will use Theorem 2.1 (iii) to prove that $\mathcal{A}$ and $\mathcal{B}$ entangle one another. To this end, take an integral extreme point $x \in Q(\mathcal{A})$ such that $x \notin Q(\mathcal{B})$, and a point $y \in Q(\mathcal{B})$. Since $Q(\mathcal{A}) \cup Q(\mathcal{B})$ is convex, $(x, y) \subseteq Q(\mathcal{A}) \cup Q(\mathcal{B})$. However, as $x \notin Q(\mathcal{B})$ and $Q(\mathcal{B})$ is a closed set, there exists an $\epsilon \in (0, \frac{1}{2}]$ such that $(1-\epsilon)x + \epsilon y \notin Q(\mathcal{B})$, in turn implying that $(1-\epsilon)x + \epsilon y \in Q(\mathcal{A})$. Thus, by Theorem 2.1 (iii), $\mathcal{A}$ entangles $\mathcal{B}$. Similarly, $\mathcal{B}$ entangles $\mathcal{A}$, and so $\mathcal{A}, \mathcal{B}$ are tangled. $\square$

We will show that the converse of this theorem holds as long as one of $\mathcal{A}, \mathcal{B}$ is ideal. The following result is needed:

Theorem 2.3 ([3] and [4], Lemma 1, Theorems 3 and 4)**.** *Let P, Q be two polyhedra in the same space. Then the following statements are equivalent:*

- *$P \cup Q$ is convex,*
- *for each extreme point $x \in P$ and each extreme point $y \in Q$, we have $(x, y) \cap (P \cup Q) \neq \emptyset$.*

Moreover, if $P \cup Q$ is convex, then it is a polyhedron, every facet of $P \cup Q$ is a facet of one of P, Q, and every vertex of $P \cup Q$ is a vertex of one of P, Q.

We are now ready to prove that,

Proposition 2.4. *If $\mathcal{A}$ is ideal and entangles $\mathcal{B}$, then $Q(\mathcal{A}) \cup Q(\mathcal{B})$ is convex.*

Proof. Take an extreme point x of $Q(\mathcal{A})$ and an extreme point y of $Q(\mathcal{B})$. By Theorem 2.3, it suffices to show that $(x, y) \cap \big(Q(\mathcal{A}) \cup Q(\mathcal{B})\big) \neq \emptyset$. As $\mathcal{A}$ is ideal, it follows that x is integral. If $x \in Q(\mathcal{B})$, then $(x, y) \subseteq Q(\mathcal{B})$. Otherwise, since $\mathcal{A}$ entangles $\mathcal{B}$, Proposition 2.1 (ii) implies that $\frac{1}{2}(x + y) \in Q(A)$. Either way, we see that $(x, y) \cap \big(Q(\mathcal{A}) \cup Q(\mathcal{B})\big) \neq \emptyset$, as required. $\square$

As an immediate corollary,

Theorem 2.5. *If $\mathcal{A},\mathcal{B}$ are tangled clutters and $Q(\mathcal{A})\cup Q(\mathcal{B})$ is not convex, then $\mathcal{A}$ and $\mathcal{B}$ are non-ideal.*

For instance, the two clutters $\mathcal{A}=\big\{\{1,2\},\{1,3,4\},\{2,3,4\}\big\}$ and $\mathcal{B}=\big\{\{3,4\},\{1,2,3\},\{1,2,4\}\big\}$ are tangled, since $b(\mathcal{A})$ and $b(\mathcal{B})$ are tangled clutters by Proposition 1.4. However, $Q(\mathcal{A})\cup Q(\mathcal{B})$ is not a convex set, since $x=\big(\frac{1}{2},\frac{1}{2},\frac{1}{2},0\big)\in Q(\mathcal{A})$ and $y=\big(0,\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)\in Q(\mathcal{B})$ but $\frac{1}{2}(x+y)\notin Q(\mathcal{A})\cup Q(\mathcal{B})$. Thus, the preceding theorem implies that both $\mathcal{A},\mathcal{B}$ are non-ideal, which is indeed the case as $\mathcal{A}/4\cong\mathcal{B}/1\cong\Delta_3$.

Finally, Theorems 2.2 and 2.5 reveal a connection between entanglement and convex union. The following proposition complements these results:

Proposition 2.6. *Let $\mathcal{A},\mathcal{B}$ be clutters over the same ground set. Then $Q(\mathcal{A})\cup Q(\mathcal{B})$ is convex if, and only if, $Q(\mathcal{A})\cup Q(\mathcal{B})=Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$.*

Proof. If $Q(\mathcal{A})\cup Q(\mathcal{B})=Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$, then $Q(\mathcal{A})\cup Q(\mathcal{B})$ is obviously convex. Conversely, assume that $Q(\mathcal{A})\cup Q(\mathcal{B})$ is convex. By Theorem 2.3, $Q(\mathcal{A})\cup Q(\mathcal{B})$ is a polyhedron, and to show that $Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))\subseteq Q(\mathcal{A})\cup Q(\mathcal{B})$, we will prove that every facet of $Q(\mathcal{A})\cup Q(\mathcal{B})$ defines a valid inequality of $Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$. Suppose $a^\top x\geq b$ defines a facet of $Q(\mathcal{A})\cup Q(\mathcal{B})$. By Theorem 2.3, $a^\top x\geq b$ also defines a facet of one of $Q(\mathcal{A}),Q(\mathcal{B})$. By symmetry, we may assume that $a^\top x\geq b$ defines a facet of $Q(\mathcal{A})$. If $a^\top x\geq b$ is a non-negativity inequality, then it is clearly valid for $Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$. Otherwise, $a^\top x\geq b$ is equivalent to $x(A)\geq 1$ for some $A\in\mathcal{A}$. As $x(A)\geq 1$ is also valid for $Q(\mathcal{B})$, it follows that $|A\cap B'|\geq 1$ for every $B'\in b(\mathcal{B})$. Thus, A is a cover of $b(\mathcal{B})$, so A contains a member of $\mathcal{B}=b(b(\mathcal{B}))$. This means that $A\in\mathrm{LCL}(\mathcal{A},\mathcal{B})$ and so $x(A)\geq 1$, and therefore $a^\top x\geq b$, is valid for $Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$. Thus, every facet of $Q(\mathcal{A})\cup Q(\mathcal{B})$ defines a valid inequality of $Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$, implying in turn that $Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))\subseteq Q(\mathcal{A})\cup Q(\mathcal{B})$. By Proposition 1.6 (3), $Q(\mathcal{A})\cup Q(\mathcal{B})\subseteq Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$, so $Q(\mathcal{A})\cup Q(\mathcal{B})=Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$, as required. $\square$

3 Opposite elements and split joins

Here we lay the groundwork for an excluded-minor characterization of entanglement. Let $\mathcal{C}$ be a clutter. We say that distinct elements e,f are *opposite* if $\{e,f\}$ is not contained in a member or a minimal cover. Opposite elements were introduced and studied in [2].

Theorem 3.1 ([2])**.** *Let $\mathcal{C}$ be a clutter and take distinct elements e,f. Then e,f are opposite if, and only if, the following statement holds:*

> ($\diamond$) *for (possibly equal) members C_e,C_f such that $e\in C_e$ and $f\in C_f$, $(C_e\cup C_f)-\{e,f\}$ contains a member.*

Proof. ($\Rightarrow$) Take members C_e,C_f such that $e\in C_e$ and $f\in C_f$. Since $\{e,f\}$ is not contained in a member, it follows that $C_e\cap\{e,f\}=\{e\}$ and $C_f\cap\{e,f\}=\{f\}$. Suppose for a contradiction that $(C_e\cup C_f)-\{e,f\}$ does not contain a member. Then the complement of $(C_e\cup C_f)-\{e,f\}$ is a cover, so it contains a minimal cover B. Note that $B \cap C_e \subseteq \{e\}$ and $B \cap C_f \subseteq \{f\}$, and as B is a cover, we must have that $\{e, f\} \subseteq B$, a contradiction as $\{e, f\}$ is not contained in a minimal cover. ($\Leftarrow$) Assume that ($\diamond$) holds. If $\{e, f\}$ is contained in a member C, then $C_e := C$ and $C_f := C$ contradict ($\diamond$). Thus, $\{e, f\}$ is not contained in a member. Suppose for a contradiction that $\{e, f\}$ is contained in a minimal cover B. Since $B - \{e\}$ is not a cover, there is a member C_e such that $C_e \cap B = \{e\}$, and since $B - \{f\}$ is not a cover, there is a member C_f such that $C_f \cap B = \{f\}$. However, as $(C_e \cup C_f) - \{e, f\}$ is disjoint from B, it cannot contain a member of $\mathcal{C}$, a contradiction to ($\diamond$). Thus, e and f are opposite. $\square$

There is a constructive way to define opposite elements. Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set E, and take new element labels $e, f \notin E$. Denote by $\mathcal{A}^e \sqcup \mathcal{B}^f$ the clutter over ground set $E \cup \{e, f\}$ whose members are the minimal sets of

$$\big\{\{e\} \cup A : A \in \mathcal{A}\big\} \cup \big\{\{f\} \cup B : B \in \mathcal{B}\big\} \cup \mathrm{LCL}(\mathcal{A}, \mathcal{B});$$

we call this clutter a *split join of $\mathcal{A}$ and $\mathcal{B}$*. Notice that $\mathcal{A}^e \sqcup \mathcal{B}^f = \mathcal{B}^f \sqcup \mathcal{A}^e$.

Proposition 3.2. *Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground E, and take new element labels e, f. Then the following statements hold:*

(1) e, f are opposite in $\mathcal{A}^e \sqcup \mathcal{B}^f$,

(2) $b(\mathcal{A}^e \sqcup \mathcal{B}^f) = b(\mathcal{A})^f \sqcup b(\mathcal{B})^e$,

(3) $(\mathcal{A}^e \sqcup \mathcal{B}^f)/e \setminus f = \mathcal{A}$, $(\mathcal{A}^e \sqcup \mathcal{B}^f) \setminus e/f = \mathcal{B}$, $(\mathcal{A}^e \sqcup \mathcal{B}^f)/e/f = \mathrm{GCP}(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}^e \sqcup \mathcal{B}^f) \setminus e \setminus f = \mathrm{LCL}(\mathcal{A}, \mathcal{B})$,

(4) for disjoint $I, J \subseteq E$, $(\mathcal{A}^e \sqcup \mathcal{B}^f) \setminus I/J = (\mathcal{A} \setminus I/J)^e \sqcup (\mathcal{B} \setminus I/J)^f$.

Proof. **(1)** By definition, no member of $\mathcal{A}^e \sqcup \mathcal{B}^f$ contains both e, f. Choose members C_e, C_f of $\mathcal{A}^e \sqcup \mathcal{B}^f$ such that $e \in C_e$ and $f \in C_f$. By Theorem 3.1, it suffices to show that $(C_e \cup C_f) - \{e, f\}$ contains a member of $\mathcal{A}^e \sqcup \mathcal{B}^f$. By construction, $C_e = \{e\} \cup A$ and $C_f = \{f\} \cup B$ for some $A \in \mathcal{A}$ and $B \in \mathcal{B}$. So $(C_e \cup C_f) - \{e, f\} = A \cup B$, and $A \cup B$ contains a member of $\mathrm{LCL}(\mathcal{A}, \mathcal{B})$, so $(C_e \cup C_f) - \{e, f\}$ contains a member of $\mathcal{A}^e \sqcup \mathcal{B}^f$, as required. **(2)** We leave it as an easy exercise for the reader to check that every member of $b(\mathcal{A})^f \sqcup b(\mathcal{B})^e$ is a cover of $\mathcal{A}^e \sqcup \mathcal{B}^f$. It remains to show that every minimal cover of $\mathcal{A}^e \sqcup \mathcal{B}^f$ contains a member of $b(\mathcal{A})^f \sqcup b(\mathcal{B})^e$. To this end, let K be a minimal cover of $\mathcal{A}^e \sqcup \mathcal{B}^f$. By (1), $|K \cap \{e, f\}| \leq 1$. If $K \cap \{e, f\} = \emptyset$, then K is a cover of $\mathcal{A}$ as well as a cover of $\mathcal{B}$, so it is also a cover of $\mathrm{GCP}(\mathcal{A}, \mathcal{B})$. In this case, by Proposition 1.6 (1), K contains a member of $\mathrm{LCL}(b(\mathcal{A}), b(\mathcal{B}))$, so K contains a member of $b(\mathcal{A})^f \sqcup b(\mathcal{B})^e$. If $K \cap \{e, f\} = \{e\}$, then $K - \{e\}$ induces a cover of $\mathcal{B}$, so $K - \{e\}$ contains a member of $b(\mathcal{B})$, implying in turn that K contains a member of $b(\mathcal{A})^f \sqcup b(\mathcal{B})^e$. Otherwise, $K \cap \{e, f\} = \{f\}$, and a similar argument tells us that K contains a member of $b(\mathcal{A})^f \sqcup b(\mathcal{B})^e$. As a result, every minimal cover of $\mathcal{A}^e \sqcup \mathcal{B}^f$ contains a member of $b(\mathcal{A})^f \sqcup b(\mathcal{B})^e$, as required. **(3)** is straight-forward. **(4)** follows from Proposition 1.6 (4). $\square$

Perhaps not surprisingly, all opposite elements arise in this manner:

Theorem 3.3. *Every clutter with opposite elements is the split join of two proper minors. In particular, given a clutter $\mathcal{C}$ and opposite elements e, f, we have $\mathcal{C} = \mathcal{A}^e \sqcup \mathcal{B}^f$, where $\mathcal{A} = \mathcal{C} \setminus f/e$ and $\mathcal{B} = \mathcal{C} \setminus e/f$.*

Proof. Let C be a member of $\mathcal{C}$. If $C \cap \{e, f\} = \{e\}$, then $C - \{e\} \in \mathcal{C} \setminus f/e = \mathcal{A}$, so C contains a member of $\mathcal{A}^e \sqcup \mathcal{B}^f$. Similarly, if $C \cap \{e, f\} = \{f\}$, then C contains a member of $\mathcal{A}^e \sqcup \mathcal{B}^f$. Otherwise, $C \cap \{e, f\} = \emptyset$, and so C contains a member of $\mathcal{A}$ as well as a member of $\mathcal{B}$, so C contains a member of $\mathrm{LCL}(\mathcal{A}, \mathcal{B})$ and therefore of $\mathcal{A}^e \sqcup \mathcal{B}^f$. Thus, every member of $\mathcal{C}$ contains a member of $\mathcal{A}^e \sqcup \mathcal{B}^f$. A similar argument applied to blockers implies that every member of $b(\mathcal{C})$ contains a member of

$$(b(\mathcal{C}) \setminus f/e)^e \sqcup (b(\mathcal{C}) \setminus e/f)^f = b(\mathcal{B})^e \sqcup b(\mathcal{A})^f = b(\mathcal{A}^e \sqcup \mathcal{B}^f);$$

Proposition 3.2 (2) is used here. Thus every minimal cover of $\mathcal{C}$ is a cover of $\mathcal{A}^e \sqcup \mathcal{B}^f$. Hence, $\mathcal{C} = \mathcal{A}^e \sqcup \mathcal{B}^f$. □

Having defined split joins, let us define a closely related operator. Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set E, and take a new element label $e \notin E$. Denote by $\mathcal{A} \vee_e \mathcal{B}$ the clutter over ground set $E \cup \{e\}$ whose members are the minimal sets of

$$\big\{\{e\} \cup C : C \in \mathrm{GCP}(\mathcal{A}, \mathcal{B})\big\} \cup \mathrm{LCL}(\mathcal{A}, \mathcal{B});$$

we call this clutter a *join of $\mathcal{A}$ and $\mathcal{B}$*. Note that $\mathcal{A} \vee_e \mathcal{B} = \mathcal{B} \vee_e \mathcal{A}$. (We will drop the subscript e whenever there is no ambiguity.) Observe that the clutter obtained from the split join $\mathcal{A}^e \sqcup \mathcal{B}^f$ after identifying e, f is simply the join $\mathcal{A} \vee \mathcal{B}$. It is worth pointing out that $\vee$ is the operator that reconstructs a clutter starting from its minors:

Remark 3.4. *Let $\mathcal{C}$ be a clutter, and take an element e. Then $(\mathcal{C} \setminus e) \vee (\mathcal{C}/e) = \mathcal{C}$.*

Moreover,

Remark 3.5. *Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set E, and let e be a new element label. Then the following statements hold:*

(1) $(\mathcal{A} \vee \mathcal{B})/e = \mathrm{GCP}(\mathcal{A}, \mathcal{B})$ *and* $(\mathcal{A} \vee \mathcal{B}) \setminus e = \mathrm{LCL}(\mathcal{A}, \mathcal{B})$*,*

(2) $b(\mathcal{A} \vee \mathcal{B}) = b(\mathcal{A}) \vee b(\mathcal{B})$*,*

(3) for disjoint $I, J \subseteq E$, $(\mathcal{A} \vee \mathcal{B}) \setminus I/J = (\mathcal{A} \setminus I/J) \vee (\mathcal{B} \setminus I/J)$.

Proof. **(1)** follows immediately from construction. **(2)** follows from Proposition 1.6 (1). **(3)** follows from Proposition 1.6 (4). □

An immediate consequence of (2), used later on in the paper, is the following:

Corollary 3.6. *Given a clutter $\mathcal{C}$, the join $\mathcal{C} \vee b(\mathcal{C})$ is equal to its blocker.*

4 An excluded-minor characterization of entanglement

Take an integer $n \geq 2$. A clutter over ground set $\{e, f, 1, \ldots, n\}$ is (e, f)-*special* if its members are

$$\big\{\{e, 1\}, \{f, 2, \ldots, n\}, \{1, 2\}, \ldots, \{1, n\}\big\}.$$

We will need the following result:

Theorem 4.1 ([2], Theorem 2.5)**.** *Let $\mathcal{C}$ be a clutter with opposite elements e, f. Then the following statements are equivalent:*

(i) there exist $C_e \in \mathcal{C}$ and $C'_f \in b(\mathcal{C})$ such that $e \in C_e$, $f \in C'_f$ and $|C_e \cap C'_f| = 1$,

(ii) $\mathcal{C}$ has an (e, f)-special minor.

Using this result, we prove the following excluded-minor characterization of entanglement:

Theorem 4.2. *Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set E. Then the following statements are equivalent:*

(i) $\mathcal{A}$ does not entangle $\mathcal{B}$,

(ii) for new elements $e, f \notin E$, $\mathcal{A}^e \sqcup \mathcal{B}^f$ has an (e, f)-special minor,

(iii) for an integer $n \geq 2$ and a partition $I \cup J \cup \{1, 2 \ldots, n\} = E$,

$$\mathcal{A} \setminus I/J = \big\{\{1\}\big\} \quad \textit{and} \quad \mathcal{B} \setminus I/J = \begin{cases} \big\{\{2\}\big\} & \textit{if } n = 2 \\ \Delta_n & \textit{if } n \geq 3. \end{cases}$$

Proof. **(i) ⇒ (ii)**: Pick $A \in \mathcal{A}$ and $A' \in b(\mathcal{A})$ such that $|A \cap A'| = 1$, A does not contain a member of $\mathcal{B}$, and A' does not contain a member of $b(\mathcal{B})$. Let $C_e := \{e\} \cup A$ and $C'_f := \{f\} \cup A'$. Clearly, C_e contains a member of $\mathcal{A}^e \sqcup \mathcal{B}^f$. As A does not contain a member of $\mathcal{B}$, we get that A does not contain a member of $\mathrm{LCL}(\mathcal{A}, \mathcal{B})$, implying in turn that $C_e \in \mathcal{A}^e \sqcup \mathcal{B}^f$. Similarly, $C'_f \in b(\mathcal{A})^f \sqcup b(\mathcal{B})^e = b(\mathcal{A}^e \sqcup \mathcal{B}^f)$. Since e, f are opposite elements of $\mathcal{A}^e \sqcup \mathcal{B}^f$ by Proposition 3.2 (1), it follows from Theorem 4.1 that $\mathcal{A}^e \sqcup \mathcal{B}^f$ has an (e, f)-special minor, so (ii) holds. **(ii) ⇒ (iii)**: Choose disjoint $I, J \subseteq E$ such that

$$(\mathcal{A}^e \sqcup \mathcal{B}^f) \setminus I/J = \big\{\{e, 1\}, \{f, 2, \ldots, n\}, \{1, 2\}, \ldots, \{1, n\}\big\},$$

for some integer $n \geq 2$. By Proposition 3.2 parts (3) and (4),

$$\mathcal{A} \setminus I/J = \big[(\mathcal{A} \setminus I/J)^e \sqcup (\mathcal{B} \setminus I/J)^f\big] \setminus f/e = \big[(\mathcal{A}^e \sqcup \mathcal{B}^f) \setminus I/J\big] \setminus f/e = \big\{\{1\}\big\}$$

and similarly,

$$\mathcal{B} \setminus I/J = \big[(\mathcal{A}^e \sqcup \mathcal{B}^f) \setminus I/J\big] \setminus e/f = \begin{cases} \big\{\{2\}\big\} & \text{if } n = 2 \\ \Delta_n & \text{if } n \geq 3, \end{cases}$$

so (iii) holds. **(iii) ⇒ (i)**: As clutters over ground set $\{1, \ldots, n\}$, $\mathcal{A} \setminus I/J$ does not entangle $\mathcal{B} \setminus I/J$: $\{1\}$ is both a member and a minimal cover of $\mathcal{A} \setminus I/J$ and $|\{1\} \cap \{1\}| = 1$, but $\{1\}$ does not contain a member of and is not a cover of $\mathcal{B} \setminus I/J$. It therefore follows from Remark 1.5 that $\mathcal{A}$ does not entangle $\mathcal{B}$, so (i) holds. □

As an immediate consequence,

Theorem 4.3. *Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set E. Then one of the following statements hold:*

(i) *$\mathcal{A}$ entangles $\mathcal{B}$,*

(ii) *$\mathcal{B}$ has a delta minor, or*

(iii) *there exists a partition $\{1,2\} \cup I \cup J = E$ such that $\mathcal{A} \setminus I/J = \big\{\{1\}\big\}$ and $\mathcal{B} \setminus I/J = \big\{\{2\}\big\}$.*

This result has the following corollary, which may be viewed as the analogue of Proposition 2.4 for delta-freeness:

Corollary 4.4. *If $\mathcal{A}$ is delta free and entangles $\mathcal{B}$, then $\mathcal{B}$ entangles $\mathcal{A}$.*

Proof. Let $\mathcal{A}, \mathcal{B}$ be clutters over the same ground set E, where $\mathcal{A}$ is delta free and entangles $\mathcal{B}$. Suppose for a contradiction that $\mathcal{B}$ does not entangle $\mathcal{A}$. Since $\mathcal{A}$ has no delta minor, it follows from Theorem 4.3 that there is a partition $\{1,2\} \cup I \cup J = E$ such that $\mathcal{A} \setminus I/J = \big\{\{1\}\big\}$ and $\mathcal{B} \setminus I/J = \big\{\{2\}\big\}$. But then by Theorem 4.2, $\mathcal{A}$ does not entangle $\mathcal{B}$, a contradiction. $\square$

The reader may wonder whether, as in Proposition 2.4, one can make the stronger conclusion that $Q(\mathcal{A}) \cup Q(\mathcal{B})$ is convex? The answer is no. For example, consider the clutter $\mathcal{C}$ whose incidence matrix is

$$M(\mathcal{C}) = \begin{pmatrix} & 1 & 1 & 1 & 1 & & \\ & & 1 & 1 & 1 & 1 & \\ & & & 1 & 1 & 1 & 1 \\ 1 & & & & 1 & 1 & 1 \\ 1 & 1 & & & & 1 & 1 \\ 1 & 1 & 1 & & & & 1 \end{pmatrix}.$$

Notice that $\mathcal{C}$ has no cover of cardinality one, and that $\{1,2,3,4\}, \{5,6,7\}$ are covers of it that partition the ground set. It therefore follows from Remark 1.4 that the clutters $\mathcal{A}, \mathcal{B}$ with incidence matrices

$$M(\mathcal{A}) = \begin{pmatrix} 1 & 1 & 1 & 1 & & & \\ & 1 & 1 & 1 & 1 & & \\ & & 1 & 1 & 1 & 1 & \\ & & & 1 & 1 & 1 & 1 \\ 1 & & & & 1 & 1 & 1 \\ 1 & 1 & & & & 1 & 1 \\ 1 & 1 & 1 & & & & 1 \end{pmatrix} \quad \text{and} \quad M(\mathcal{B}) = \begin{pmatrix} & & & & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & & \\ & & 1 & 1 & 1 & 1 & \\ 1 & 1 & & & & 1 & 1 \\ 1 & 1 & 1 & & & & 1 \end{pmatrix}$$

are tangled. It can be readily checked that these clutters are delta free, as $\mathcal{A}$ is mni and not a delta, and the only mni minor of $\mathcal{B}$ is $\mathcal{B} \setminus \{1,4\}$ which is not a delta. Take $x^\star := (1\,0\,0\,1\,0\,0\,0) \in Q(\mathcal{A})$ and $y^\star := \left(0\,\frac{1}{3}\,\frac{1}{3}\,0\,\frac{1}{3}\,\frac{1}{3}\,\frac{1}{3}\right) \in Q(\mathcal{B})$. Then, for any $\lambda \in (0, \frac{1}{4})$, $\lambda x^\star + (1-\lambda) y^\star \notin Q(\mathcal{A}) \cup Q(\mathcal{B})$, implying in turn that $Q(\mathcal{A}) \cup Q(\mathcal{B})$ is not convex.

5 Entanglement of binary clutters

Here we prove the following statement:

Theorem 5.1. *Let $\mathcal{A}, \mathcal{B}$ be binary clutters where $\mathcal{A}$ entangles $\mathcal{B}$. Then one of $\mathcal{A}, \mathcal{B}$ is a lift of the other.*

Let $\mathbb{P}_4 := \big\{\{1,3\},\{2,4\},\{3,4\}\big\}$. It can be readily checked that $\mathbb{P}_4$ is a non-binary clutter and it is isomorphic to its blocker. Together with the deltas, $\mathbb{P}_4$ gives the class of excluded-minors defining binary clutters:

Theorem 5.2 ([20])**.** *For a clutter $\mathcal{C}$, the following statements are equivalent:*

- *$\mathcal{C}$ is not binary,*
- *for some $C \in \mathcal{C}$ and $B \in b(\mathcal{C})$, $|C \cap B| = 2$,*
- *$\mathcal{C}$ has one of $\mathbb{P}_4, \{\Delta_n : n \geq 3\}$ as a minor.*

As a consequence of the preceding two results,

Corollary 5.3. *Let $\mathcal{A}, \mathcal{B}$ be clutters one of which entangles the other but neither of which is a lift of the other. Then one of $\mathcal{A}, \mathcal{B}$ has one of $\mathbb{P}_4, \{\Delta_n : n \geq 3\}$ as a minor.*

A couple of tools need to be developed before we can prove Theorem 5.1.

5.1 Delta free clutters

We will need the following result:

Theorem 5.4 ([1])**.** *Let $\mathcal{C}$ be a clutter and take an element e. If there are distinct members C_1, C_2, C_3 such that $e \in C_1 \cap C_2$, $e \notin C_3$ and $C_1 \cup C_2 \subseteq \{e\} \cup C_3$, then $\mathcal{C}$ has a delta minor using e.*

Using this theorem, we prove the following:

Theorem 5.5. *Let $\mathcal{A}, \mathcal{B}$ be tangled clutters. Then $\mathcal{A}$ and $\mathcal{B}$ are delta free if, and only if, $\mathrm{GCP}(\mathcal{A}, \mathcal{B})$ and $\mathrm{LCL}(\mathcal{A}, \mathcal{B})$ are delta free.*

Proof. We will need the following two claims:

Claim 1. *Take an integer $n \geq 3$. If $\mathcal{A} = \Delta_n$ or $\mathcal{B} = \Delta_n$, then one of $\mathrm{GCP}(\mathcal{A}, \mathcal{B}), \mathrm{LCL}(\mathcal{A}, \mathcal{B})$ has a delta minor.*

Proof of Claim. By symmetry, we may assume that $\mathcal{A} = \Delta_n$, that is,

$$\mathcal{A} = \big\{\{2,3,\ldots,n\},\{1,2\},\{1,3\},\ldots,\{1,n\}\big\}.$$

If $\mathcal{A}$ is a lift of $\mathcal{B}$, then $\big\{\mathrm{LCL}(\mathcal{A}, \mathcal{B}), \mathrm{GCP}(\mathcal{A}, \mathcal{B})\big\} = \big\{\mathcal{A}, \mathcal{B}\big\}$ and so we are done. Otherwise, by Proposition 1.2, $\mathcal{B}$ does not have a member of cardinality 1 (for if not, then every cover of $\mathcal{B}$ is also a cover of $\mathcal{A}$), and there is a member $A \in \mathcal{A}$ that does not contain a member of $\mathcal{B}$. Since $A \in b(\mathcal{A}) = \mathcal{A}$ and A intersects every other member of $\mathcal{A}$ once, we get that every member of $\mathcal{A}$ other than A contains a member of $\mathcal{B}$, as $\mathcal{A}$ entangles $\mathcal{B}$. After a relabeling, if necessary, we may assume that either $A = \{2, 3, \ldots, n\}$ or $A = \{1, n\}$ (these two cases are the same when $n = 3$).

Assume in the first case that $A = \{2, 3, \ldots, n\}$. As $\mathcal{B}$ has no member of cardinality 1, it follows that $\{1, 2\}$, $\{1, 3\}, \ldots, \{1, n\} \in \mathcal{B}$. For A does not contain a member of $\mathcal{B}$, we have that $\mathcal{B} = \big\{\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}\big\}$, implying by Proposition 1.2 that $\mathcal{B}$ is a lift of $\mathcal{A}$, so $\big\{\mathrm{LCL}(\mathcal{A}, \mathcal{B}), \mathrm{GCP}(\mathcal{A}, \mathcal{B})\big\} = \big\{\mathcal{A}, \mathcal{B}\big\}$ and we are done.

Assume in the remaining case that $A = \{1, n\}$ and $n \geq 4$. As $\mathcal{B}$ has no member of cardinality 1, $\{1, 2\}$, $\{1, 3\}, \ldots, \{1, n-1\} \in \mathcal{B}$ and there is a member $B \in \mathcal{B}$ of cardinality at least 2 such that $B \subseteq \{2, 3, \ldots, n\}$. Clearly, $\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}, B \in \mathrm{GCP}(\mathcal{A}, \mathcal{B})$. If $\{i, j\} \subseteq B$, then by Theorem 5.4, the members $\{1, i\}, \{1, j\}, B$ yield a delta minor in $\mathrm{GCP}(\mathcal{A}, \mathcal{B})$, as claimed. $\Diamond$

It therefore follows from Proposition 1.6 (4) that if $\mathrm{GCP}(\mathcal{A}, \mathcal{B}), \mathrm{LCL}(\mathcal{A}, \mathcal{B})$ are delta free, then so are $\mathcal{A}, \mathcal{B}$.

Claim 2. *Take an integer $n \geq 3$. If $\mathrm{GCP}(\mathcal{A}, \mathcal{B}) = \Delta_n$ or $\mathrm{LCL}(\mathcal{A}, \mathcal{B}) = \Delta_n$, then one of $\mathcal{A}, \mathcal{B}$ has a delta minor.* [6]

Proof of Claim. By Proposition 1.6 (1) and the equation $b(\Delta_n) = \Delta_n$, it suffices to show that if $\mathrm{GCP}(\mathcal{A}, \mathcal{B}) = \Delta_n$, then one of $\mathcal{A}, \mathcal{B}$ has a delta minor. If one of $\mathcal{A}, \mathcal{B}$ is a lift of the other, then we are done as $\big\{\mathcal{A}, \mathcal{B}\big\} = \big\{\mathrm{LCL}(\mathcal{A}, \mathcal{B}), \mathrm{GCP}(\mathcal{A}, \mathcal{B})\big\}$. Otherwise, neither $\mathcal{A}$ nor $\mathcal{B}$ is a lift of the other, and so in particular, neither of them has a cover of cardinality 1. Suppose

$$\mathrm{GCP}(\mathcal{A}, \mathcal{B}) = \big\{\{2, 3, \ldots, n\}, \{1, 2\}, \{1, 3\}, \ldots, \{1, n\}\big\}.$$

Then by Proposition 1.6 (2),

$$\mathbb{R}^n \ni x^\star := \begin{pmatrix} \frac{n-2}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \end{pmatrix} \in Q(\mathrm{GCP}(\mathcal{A}, \mathcal{B})) = Q(\mathcal{A}) \cap Q(\mathcal{B}).$$

In particular, for each member L of $\mathcal{A}$ or $\mathcal{B}$, either

- $1 \in L$ and $L \cap \{2, \ldots, n\} \neq \emptyset$, or
- $L = \{2, 3, \ldots, n\}$.

We claim that either $\mathcal{A}$ or $\mathcal{B}$ has two members of the first type and a member of the second type. Suppose otherwise. If every member of $\mathcal{B}$ contains 1, then $\{1\}$ is a cover of $\mathcal{B}$, which is not the case. Hence, $\{2, 3, \ldots, n\} \in \mathcal{B}$ and similarly, $\{2, 3, \ldots, n\} \in \mathcal{A}$. In particular, since $\mathrm{GCP}(\mathcal{A}, \mathcal{B}) = \Delta_n$ and neither of $\mathcal{A}, \mathcal{B}$ contains two members of the first type, we get that $n = 3$ and

$$\mathcal{A} = \big\{\{2, 3\}, \{1, 2\}\big\} \quad \text{and} \quad \mathcal{B} = \big\{\{2, 3\}, \{1, 3\}\big\}.$$

However, $\mathcal{A}$ does not entangle $\mathcal{B}$, a contradiction. Thus, one of $\mathcal{A}$ and $\mathcal{B}$, say $\mathcal{A}$, has two members of the first type and a member of the second type. By Theorem 5.4 then, $\mathcal{A}$ has a delta minor, as claimed. $\Diamond$

[6] Using heavier artillery we will show in Proposition 7.1 of §7 that one of $\mathcal{A}, \mathcal{B}$ is Δ_n.

Thus, by Proposition 1.6 (4), if $\mathcal{A}, \mathcal{B}$ are delta free, then so are $\mathrm{GCP}(\mathcal{A},\mathcal{B}), \mathrm{LCL}(\mathcal{A},\mathcal{B})$, thereby finishing the proof of the theorem. $\square$

Before moving on to the next tool, let us quickly state an application of the results obtained so far. Given a clutter $\mathcal{L}$ with opposite elements $e, f \in E(\mathcal{L})$, the family

$$\big\{L : L \in \mathcal{L}, f \notin L\big\} \cup \big\{L \triangle \{e,f\} : L \in \mathcal{L}, f \in L\big\}$$

is a clutter over ground set $E(\mathcal{L}) - \{f\}$ and called a *single identification of* $\mathcal{L}$ ([2], Proposition 3.2). A clutter $\mathcal{C}$ obtained from $\mathcal{L}$ after applying a series of single identifications is an *identification of* $\mathcal{L}$, and inversely, $\mathcal{L}$ is called a *split of* $\mathcal{C}$. It is known that splitting preserves idealness, the packing property, as well as the max-flow min-cut property [2].[7] We will show below that splitting also preserves delta-free-ness:

Corollary 5.6. *If a clutter is delta free, then so is every split of it.*

Proof. It suffices to prove this for single splits. By Theorem 3.3, it suffices to prove the following:

> Take clutters $\mathcal{A}, \mathcal{B}$ over the same ground set such that $\mathcal{A} \vee_e \mathcal{B}$ is delta free. Then $\mathcal{A}^e \sqcup \mathcal{B}^f$ is also delta free.

To this end, assume for a contradiction that $\mathcal{A} \vee_e \mathcal{B}$ is delta free and $\mathcal{A}^e \sqcup \mathcal{B}^f$ has a delta minor. In particular, $\mathcal{A} \vee_e \mathcal{B} \ncong \mathcal{A}^e \sqcup \mathcal{B}^f$, so neither of $\mathcal{A}, \mathcal{B}$ is a lift of the other. Since the join $\mathcal{A} \vee_e \mathcal{B}$ has no delta minor, the split join $\mathcal{A}^e \sqcup \mathcal{B}^f$ has no (e,f)- or (f,e)-special minors, implying by Theorem 4.2 that $\mathcal{A}, \mathcal{B}$ are tangled clutters. For $\mathcal{A} \vee \mathcal{B}$ is delta free, its minors $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ and $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ are delta free, so by Theorem 5.5, $\mathcal{A}$ and $\mathcal{B}$ are also delta free. As a result, by Proposition 3.2 (3), the delta minor of $\mathcal{A}^e \sqcup \mathcal{B}^f$ uses at least one of e, f. In fact, as a delta does not have opposite elements, it follows from Proposition 3.2 (1) that the delta minor of $\mathcal{A}^e \sqcup \mathcal{B}^f$ uses precisely one of e, f. We may assume that $\mathcal{A}^e \sqcup \mathcal{B}^f / e = \Delta_n$ for some $n \geq 3$. So

$$\begin{aligned}\big\{\{1,2\},\{1,3\},\ldots,\{1,n\},\{2,3,\ldots,n\}\big\} = \Delta_n &= \text{minimal members of } \mathcal{A} \cup \big\{\{f\} \cup B : B \in \mathcal{B}\big\} \\ &= \mathcal{A} \cup \left\{\{f\} \cup B : \begin{array}{c} B \in \mathcal{B} \text{ where } B \text{ does not} \\ \text{contain a member of } \mathcal{A}\end{array}\right\}.\end{aligned}$$

If $f = 1$, then $\big\{\{2,3,\ldots,n\}\big\} = \mathcal{A}$, so $\mathcal{A}$ is a lift of $\mathcal{B}$, which is not the case. Thus, $f \neq 1$ and so we may assume that $f = n$. But then $\mathcal{A} = \big\{\{1,2\},\ldots,\{1,n-1\}\big\}$ and $\{1\} \in \mathcal{B}$, so $\mathcal{A}$ is a lift of $\mathcal{B}$, a contradiction. This finishes the proof of the corollary. $\square$

5.2 Opposite elements in binary clutters

A family $\mathfrak{F}$ of clutters is *split-closed* if, for each clutter $\mathcal{C} \in \mathfrak{F}$, every split of $\mathcal{C}$ has a minor in $\mathfrak{F}$. For instance, $\{\mathbb{P}_4\} \cup \{\Delta_n : n \geq 3\}$ is a split-closed family [2]. It follows from definition that,

[7] A clutter has the *max-flow min-cut property* if any clutter obtained after applying a series of replications has the packing property.

Remark 5.7 ([2])**.** *Let $\mathfrak{F}$ be a split-closed family. If a clutter $\mathcal{L}$ has no minor in $\mathfrak{F}$, then neither does any identification of $\mathcal{L}$.*

If $\mathcal{L}$ has opposite elements e, f and $\mathcal{C}$ is obtained after identifying e and f, then $b(\mathcal{C})$ is obtained from $b(\mathcal{L})$ after identifying e and f ([2], Proposition 3.2). Using this fact and the remark above, we prove the main result of this subsection:

Theorem 5.8. *Let $\mathcal{L}$ be a clutter with opposite elements e, f. If $\mathcal{L}$ is binary, then one of e, f is not contained in any member.*

Proof. Suppose for a contradiction that $\mathcal{L}$ is binary and each one of e, f is contained in a member. Let $\mathcal{C}$ be the clutter obtained from $\mathcal{L}$ after identifying e, f. As $\mathcal{L}$ is binary, it has no minor in $\mathfrak{F} := \{\mathbb{P}_4\} \cup \{\Delta_n : n \geq 3\}$. Since $\mathfrak{F}$ is a split-closed family, it follows from Remark 5.7 that $\mathcal{C}$ does not have a minor in $\mathfrak{F}$ either. Thus, by Theorem 5.2, $\mathcal{C}$ is also a binary clutter. Since both e, f are contained in members of $\mathcal{L}$, there exist $L_e \in \mathcal{L}$ and $K_f \in b(\mathcal{L})$ such that $e \in L_e$ and $f \in K_f$. Since $\mathcal{L}$ is binary, $|L_e \cap K_f|$ is odd. However, $C_e := L_e$ is a member and $B_e := K_f \triangle \{e, f\}$ is a minimal cover of $\mathcal{C}$, and

$$|C_e \cap B_e| = |L_e \cap K_f| + 1.$$

But then $|C_e \cap B_e|$ is even, implying that $\mathcal{C}$ is non-binary, a contradiction. $\square$

5.3 Proof of Theorem 5.1

Let $\mathcal{C}$ be a clutter and take distinct elements e, f. We say that $\mathcal{C}$ is *$\{e, f\}$-simple* if there is a partition of $E(\mathcal{C}) - \{e, f\}$ into non-empty parts X, Y such that

- $\{e\} \cup X, \{f\} \cup Y$ are the only members containing either e or f, and
- $\{e\} \cup Y, \{f\} \cup X$ are the only minimal covers containing either e or f.

Notice that e, f are opposite elements of an $\{e, f\}$-simple clutter, and that the blocker of an $\{e, f\}$-simple clutter is also $\{e, f\}$-simple.

Proposition 5.9 ([2], Proposition 2.4)**.** *Let $\mathcal{C}$ be a clutter with opposite elements e, f each of which is contained in a member. Then $\mathcal{C}$ has an $\{e, f\}$-simple minor.*

The following is the last needed ingredient:

Proposition 5.10. *Let $\mathcal{A}, \mathcal{B}$ be tangled binary clutters. Then* $\mathrm{GCP}(\mathcal{A}, \mathcal{B})$ *and* $\mathrm{LCL}(\mathcal{A}, \mathcal{B})$ *are binary clutters.*

Proof. By Proposition 1.6 (1) and the fact that taking blockers preserves being binary, it suffices to show that $\mathrm{GCP}(\mathcal{A}, \mathcal{B})$ is binary. Since $\mathcal{A}, \mathcal{B}$ are delta free, it follows from Theorem 5.5 that $\mathrm{GCP}(\mathcal{A}, \mathcal{B})$ is delta free. By Theorem 5.2, it suffices to show that $\mathrm{GCP}(\mathcal{A}, \mathcal{B})$ has no $\mathbb{P}_4$ minor. Suppose otherwise. By Proposition 1.6 (4), we may assume that

$$\mathrm{GCP}(\mathcal{A}, \mathcal{B}) = \mathbb{P}_4 = \big\{\{1, 3\}, \{2, 4\}, \{3, 4\}\big\}.$$

If one of $\mathcal{A},\mathcal{B}$ is a lift of the other, then one of them is $\mathbb{P}_4$, which is not the case as $\mathcal{A},\mathcal{B}$ are binary. Thus, neither of $\mathcal{A},\mathcal{B}$ is a lift of the other, and so in particular, neither of them has a cover of cardinality 1. Thus, as $\{3,4\}\in b(\mathrm{GCP}(\mathcal{A},\mathcal{B}))=\mathrm{LCL}(b(\mathcal{A}),b(\mathcal{B}))$, it follows that $\{3,4\}\in b(\mathcal{A})$ and $\{3,4\}\in b(\mathcal{B})$. Moreover, as $\{3,4\}\in \mathrm{GCP}(\mathcal{A},\mathcal{B})$, we may assume that $\{3,4\}\in\mathcal{A}$. It can now be readily checked that $\mathcal{A}$ is either isomorphic to $\mathbb{P}_4$ or has a delta minor, a contradiction as $\mathcal{A}$ is binary. This finishes the proof of the proposition. □

We are now ready to prove Theorem 5.1, stating that if a binary clutter entangles another, then one is a lift of the other:

Proof of Theorem 5.1. Let $\mathcal{A},\mathcal{B}$ be binary clutters where $\mathcal{A}$ entangles $\mathcal{B}$. Since $\mathcal{A}$ is delta free, it follows from Corollary 4.4 that $\mathcal{B}$ also entangles $\mathcal{A}$. Thus, $\mathcal{A},\mathcal{B}$ are tangled. By Proposition 5.10, $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ and $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ are binary clutters. Suppose for a contradiction that neither of $\mathcal{A},\mathcal{B}$ is a lift of the other. Take new element labels e,f and let $\mathcal{C}:=\mathcal{A}^e\sqcup\mathcal{B}^f$. By Proposition 3.2 (1), e and f are opposite elements of $\mathcal{C}$ each of which, by our contrary assumption, is contained in a member of $\mathcal{C}$. Thus, by Proposition 5.9, there exist disjoint $I,J\subseteq E(\mathcal{A})=E(\mathcal{B})$ such that $\mathcal{C}':=\mathcal{C}\setminus I/J$ is $\{e,f\}$-simple. That is, there is a partition of $E(\mathcal{C}')-\{e,f\}$ into non-empty parts X,Y such that

- $\{e\}\cup X,\{f\}\cup Y$ are the only members of $\mathcal{C}'$ containing either e or f, and
- $\{e\}\cup Y,\{f\}\cup X$ are the only minimal covers of $\mathcal{C}'$ containing either e or f.

Since $\mathcal{A},\mathcal{B}$ are tangled, it follows from Theorem 4.2 that $\mathcal{C}$, and therefore $\mathcal{C}'$, does not have an (e,f)- or an (f,e)-special minor, implying by Theorem 4.1 that $|X|>1$ and $|Y|>1$.

Claim 1. *$\mathcal{C}'\setminus e/f,\mathcal{C}'/e\setminus f,\mathcal{C}'\setminus e\setminus f,\mathcal{C}'/e/f$ are binary.*

Proof of Claim. Since $\mathcal{A},\mathcal{B},\mathrm{GCP}(\mathcal{A},\mathcal{B}),\mathrm{LCL}(\mathcal{A},\mathcal{B})$ are binary clutters, it follows from Proposition 3.2 (3) that $\mathcal{C}\setminus e/f,\mathcal{C}/e\setminus f,\mathcal{C}\setminus e\setminus f,\mathcal{C}/e/f$ are binary, implying in turn that their minors $\mathcal{C}'\setminus e/f,\mathcal{C}'/e\setminus f,\mathcal{C}'\setminus e\setminus f,\mathcal{C}'/e/f$ are binary. ◊

Claim 2. *$|X|$ and $|Y|$ are odd.*

Proof of Claim. X is both a member and a minimal cover of the clutter $\mathcal{C}'/e\setminus f$. Since this clutter is binary by Claim 1, it follows that $|X|$ is odd. Similarly, $|Y|$ is odd. ◊

In particular, $|X|\geq 3$ and $|Y|\geq 3$. Since $\mathcal{C}'$ is a clutter with opposite elements e and f each of which is contained in a member, it is non-binary by Theorem 5.8. Therefore, by Theorem 5.2, there exist $L\in\mathcal{C}'$ and $K\in b(\mathcal{C}')$ such that $|L\cap K|=2$. It follows from Claim 2 that either $L\cap\{e,f\}=\emptyset$ or $K\cap\{e,f\}=\emptyset$.

Claim 3. *$L\cap\{e,f\}=\emptyset$ and $K\cap\{e,f\}=\emptyset$.*

Proof of Claim. Suppose otherwise. By the symmetry between $\mathcal{C}'$ and its blocker, and the symmetry between e and f, we may assume that $L \cap \{e, f\} = \emptyset$ and $K = \{e\} \cup Y$. Note that $K - \{e\} = Y$ is a minimal cover of $\mathcal{C}' \setminus e/f$, and since $|Y| \geq 3$, L is a member of $\mathcal{C}' \setminus e/f$. But then $|L \cap Y| = |L \cap K| = 2$, a contradiction as $\mathcal{C}' \setminus e/f$ is binary by Claim 1. $\Diamond$

Claim 4. *L contains one of X, Y and intersects the other one exactly once. Similarly, K contains one of X, Y and intersects the other one exactly once.*

Proof of Claim. Since $K \cap \{e, f\} = \emptyset$ by Claim 3, K is a minimal cover of $\mathcal{C}'/e/f$. Since this clutter is binary by Claim 1, L is not a member of $\mathcal{C}'/e/f$, so as $L \cap \{e, f\} = \emptyset$ by Claim 3, it follows that L contains one of X, Y. Since for every $g \in L$ there is a minimal cover K_g of $\mathcal{C}'$ such that $L \cap K_g = \{g\}$, between X and Y, L intersects the one it does not contain exactly once. Similarly, K contains one of X, Y and intersects the other one exactly once. $\Diamond$

Since $|X| \geq 3$, $|Y| \geq 3$ and $|L \cap K| = 2$, it follows that L contains one of X, Y and K contains the other one. By the symmetry between e and f, we may assume that $X \subseteq L$, $L \cap Y = \{y\}$, $Y \subseteq K$ and $K \cap X = \{x\}$. Since K is a minimal cover of $\mathcal{C}'$, there is a member L' of $\mathcal{C}'$ such that $L' \cap K = \{y\}$. Since $\mathcal{C}'$ is $\{e, f\}$-simple, it follows that $L' \cap \{e, f\} = \emptyset$, implying in turn that $L' \subsetneq L$, a contradiction. $\square$

6 Greatest common projection and least common lift of tangled clutters

A key tool needed in proving the main result of the previous section was Theorem 5.5 stating that tangled clutters $\mathcal{A}$ and $\mathcal{B}$ are delta free if, and only if, $\mathrm{GCP}(\mathcal{A}, \mathcal{B})$ and $\mathrm{LCL}(\mathcal{A}, \mathcal{B})$ are delta free. Exploiting the geometry of entanglement studied in §2, we prove the following analogue for idealness:

Theorem 6.1. *Let $\mathcal{A}, \mathcal{B}$ be tangled clutters. Then $\mathcal{A}, \mathcal{B}$ are ideal if, and only if, $\mathrm{LCL}(\mathcal{A}, \mathcal{B}), \mathrm{GCP}(\mathcal{A}, \mathcal{B})$ are ideal.*

A proof is provided in §6.1. For instance, take clutters $\mathcal{A}, \mathcal{B}$ over the same ground set whose incidence matrices are

$$M(\mathcal{A}) = \left(\begin{array}{cccccc} 1 & & 1 & & & 1 \\ 1 & & & 1 & 1 & \\ & 1 & 1 & & 1 & \\ & 1 & & 1 & & 1 \\ \hline & 1 & & 1 & 1 & \end{array}\right) \quad \text{and} \quad M(\mathcal{B}) = \left(\begin{array}{cccccc} 1 & & 1 & & & 1 \\ 1 & & & 1 & 1 & \\ & 1 & 1 & & 1 & \\ & 1 & & 1 & & 1 \\ \hline & 1 & 1 & & & 1 \end{array}\right).$$

It can be readily checked that $\mathcal{A}, \mathcal{B}$ are ideal tangled clutters,

$$M(\mathrm{GCP}(\mathcal{A}, \mathcal{B})) = \left(\begin{array}{cccccc} 1 & & 1 & & & 1 \\ 1 & & & 1 & 1 & \\ & 1 & 1 & & 1 & \\ & 1 & & 1 & & 1 \\ \hline & 1 & & 1 & 1 & \\ & 1 & 1 & & & 1 \end{array}\right) \quad \text{and} \quad M(\mathrm{LCL}(\mathcal{A}, \mathcal{B})) = \begin{pmatrix} 1 & & 1 & & & 1 \\ 1 & & & 1 & 1 & \\ & 1 & 1 & & 1 & \\ & 1 & & 1 & & 1 \end{pmatrix},$$

and that $\mathrm{GCP}(\mathcal{A},\mathcal{B}),\mathrm{LCL}(\mathcal{A},\mathcal{B})$ are also ideal clutters. Notice however that $\mathcal{A},\mathcal{B},\mathrm{GCP}(\mathcal{A},\mathcal{B})$ have the packing property while the clutter $\mathrm{LCL}(\mathcal{A},\mathcal{B}) = Q_6$ does not, so the situation is different for the packing property:

Theorem 6.2. *Let $\mathcal{A},\mathcal{B}$ be tangled clutters. Then the following statements hold:*

(i) If $\mathcal{A},\mathcal{B}$ have the packing property, then $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ has the packing property.

(ii) If $\mathrm{LCL}(\mathcal{A},\mathcal{B}),\mathrm{GCP}(\mathcal{A},\mathcal{B})$ have the packing property, then so do $\mathcal{A},\mathcal{B}$.

We prove this theorem in §6.2. One reason for the difference between these two results is the geometry inherited by idealness and its lack thereof in the packing property. Another reason, which itself is a by-product of the first, is that idealness is closed under taking blockers, whereas this is not the case for the packing property. For instance, Q_6 does not have the packing property while its blocker does.[8]

6.1 Proof of Theorem 6.1

We first prove that if tangled clutters $\mathcal{A},\mathcal{B}$ are ideal, then $\mathrm{LCL}(\mathcal{A},\mathcal{B}),\mathrm{GCP}(\mathcal{A},\mathcal{B})$ are ideal as well.

Proof of Theorem 6.1 ($\Rightarrow$). We first show that,

($\star$) if tangled clutters $\mathcal{A},\mathcal{B}$ are ideal, then $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ is ideal.

To this end, take tangled clutters $\mathcal{A},\mathcal{B}$ that are ideal. It follows from Theorem 2.5 that $Q(\mathcal{A})\cup Q(\mathcal{B})$ is convex. By Theorem 2.3, every vertex of $Q(\mathcal{A})\cup Q(\mathcal{B})$ is a vertex of one of $Q(\mathcal{A}),Q(\mathcal{B})$, and so in particular, $Q(\mathcal{A})\cup Q(\mathcal{B})$ is an integral polyhedron. However, by Proposition 2.6, $Q(\mathcal{A})\cup Q(\mathcal{B}) = Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$, so $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ is an ideal clutter. This proves ($\star$). It remains to prove that

($\diamond$) if tangled clutters $\mathcal{A},\mathcal{B}$ are ideal, then $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ is ideal.

Well, take tangled clutters $\mathcal{A},\mathcal{B}$ that are ideal. Then $b(\mathcal{A}),b(\mathcal{B})$ are also tangled clutters that are ideal. It therefore follows from ($\star$) that $\mathrm{LCL}(b(\mathcal{A}),b(\mathcal{B}))$ is an ideal clutter. By Proposition 1.6 (1), $\mathrm{LCL}(b(\mathcal{A}),b(\mathcal{B})) = b(\mathrm{GCP}(\mathcal{A},\mathcal{B}))$, so $b(\mathrm{GCP}(\mathcal{A},\mathcal{B}))$ is ideal, implying in turn that $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ is an ideal clutter. This proves ($\diamond$). Together, ($\star$) and ($\diamond$) prove the ($\Rightarrow$) direction of Theorem 6.1. $\square$

To prove the converse of Theorem 6.1 we need a seminal result of Alfred Lehman mentioned in the introduction. Recall that a non-ideal clutter is minimally non-ideal (mni) if every proper minor of it is ideal. The deltas for instance are mni, and in fact, Δ_3 is the only mni clutter with at most 3 elements. For a clutter $\mathcal{C}$, denote by $\overline{\mathcal{C}}$ the clutter of its minimum cardinality members. Given an integer $r \geq 1$, a square $0,1$ matrix is *r-regular* if every row and every column has precisely r ones.

Theorem 6.3 ([17], also see [19])**.** *Let $\mathcal{K}$ be a minimally non-ideal clutter that is not a delta, $n := |E(\mathcal{K})|$ and $\mathcal{L} := b(\mathcal{K})$. Then $\mathcal{L}$ is minimally non-ideal and the following statements hold:*

[8] It is worth pointing out that the analogue of Theorem 6.2 holds for the max-flow min-cut property.

- *$M(\overline{\mathcal{K}})$ and $M(\overline{\mathcal{L}})$ are square and non-singular matrices,*
- *for some integers $r \geq 2$ and $s \geq 2$ such that $rs - n \geq 1$, $M(\overline{\mathcal{K}})$ is r-regular and $M(\overline{\mathcal{L}})$ is s-regular, and*
- *after possibly permuting the rows of $M(\overline{\mathcal{L}})$, we have*

$$M(\overline{\mathcal{K}})M(\overline{\mathcal{L}})^\top = J + (rs - n)I.$$

Here, J denotes the all-ones matrix and I the identity matrix of appropriate dimensions.

We are now ready to prove that for tangled clutters $\mathcal{A}$ and $\mathcal{B}$ one of which is non-ideal, one of $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ and $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ is also non-ideal:

Proof of Theorem 6.1 ($\Leftarrow$). Take tangled clutters $\mathcal{A},\mathcal{B}$ over ground set E (at least) one of which is non-ideal. If one of $\mathcal{A},\mathcal{B}$ has a delta minor, then by Theorem 5.5, one of $\mathrm{LCL}(\mathcal{A},\mathcal{B}),\mathrm{GCP}(\mathcal{A},\mathcal{B})$ has a delta minor and is therefore non-ideal, so we are done. We may therefore assume that both $\mathcal{A},\mathcal{B}$ are delta free. By Proposition 1.6 (4), we may assume that $\mathcal{A}$ is mni and not a delta. Let us appeal to Theorem 6.3. Let $n := |E|$ and pick integers $r, s \geq 2$ such that $rs - n \geq 1$, $M(\overline{\mathcal{A}})$ is a square non-singular r-regular matrix, $M(\overline{b(\mathcal{A})})$ is a square non-singular s-regular matrix, and after possibly permuting the rows of $M(\overline{b(\mathcal{A})})$,

$$M(\overline{\mathcal{A}})M(\overline{b(\mathcal{A})})^\top = J + (rs - n)I.$$

Let $x^\star := \left(\frac{1}{r}, \frac{1}{r}, \ldots, \frac{1}{r}\right) \in \mathbb{R}^n_+$ and $y^\star := \left(\frac{1}{s}, \frac{1}{s}, \ldots, \frac{1}{s}\right) \in \mathbb{R}^n_+$. Since every member of $\mathcal{A}$ has at least r elements and $M(\overline{\mathcal{A}})$ is non-singular, $x^\star$ is an extreme point of $Q(\mathcal{A})$. Similarly, since every member of $b(\mathcal{A})$ has at least s elements and $M(\overline{b(\mathcal{A})})$ is non-singular, $y^\star$ is an extreme point of $Q(b(\mathcal{A}))$. Observe that $x^\star \in Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$ and $y^\star \in Q(\mathrm{LCL}(b(\mathcal{A}), b(\mathcal{B})))$.

We claim that one of $Q(\mathrm{LCL}(\mathcal{A},\mathcal{B})), Q(\mathrm{LCL}(b(\mathcal{A}), b(\mathcal{B})))$ is not integral. Suppose otherwise. Since $y^\star$ cannot be an extreme point of $Q(\mathrm{LCL}(b(\mathcal{A}), b(\mathcal{B})))$, not every member of $\overline{b(\mathcal{A})}$ is a member of $\mathrm{LCL}(b(\mathcal{A}), b(\mathcal{B}))$, so there exists $A' \in \overline{b(\mathcal{A})}$ such that A' is not a cover of $\mathcal{B}$. The matrix equation above implies that A' intersects all but one member of $\overline{\mathcal{A}}$ precisely once. Since $\mathcal{A}$ entangles $\mathcal{B}$, all but one member of $\overline{\mathcal{A}}$ contain a member of $\mathcal{B}$, and subsequently, all but one member of $\overline{\mathcal{A}}$ are members of $\mathrm{LCL}(\mathcal{A},\mathcal{B})$. As a result, $x^\star$ lies on an edge of the integral polyhedron $Q(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$, so for some $\alpha \in (0,1)$ and $L_1, L_2 \in b(\mathrm{LCL}(\mathcal{A},\mathcal{B}))$,

$$x^\star = \alpha\chi_{L_1} + (1-\alpha)\chi_{L_2}.$$

This means that $r = 2$. By a similar argument, $s = 2$. Since $n \leq rs - 1$, it follows that $n \leq 3$, implying in turn that $\mathcal{A} = \Delta_3$, a contradiction as $\mathcal{A}$ is not a delta. Thus, one of $Q(\mathrm{LCL}(\mathcal{A},\mathcal{B})), Q(\mathrm{LCL}(b(\mathcal{A}), b(\mathcal{B})))$ is not integral, so one of $\mathrm{LCL}(\mathcal{A},\mathcal{B}), \mathrm{LCL}(b(\mathcal{A}), b(\mathcal{B})) = b(\mathrm{GCP}(\mathcal{A},\mathcal{B}))$ is non-ideal, and so one of $\mathrm{LCL}(\mathcal{A},\mathcal{B})$, $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ is non-ideal, as required. $\square$

6.2 Proof of Theorem 6.2

Let $\mathcal{C}$ be an ideal clutter over ground set E. Consider the dual pair of linear programs

$$(\mathrm{P})\begin{cases} \min & \sum(x_g : g\in E) \\ \text{s.t.} & \sum(x_g : g \in C)\geq 1 \quad \forall\, C\in\mathcal{C} \\ & x\geq \mathbf{0} \end{cases} \qquad (\mathrm{D})\begin{cases} \max & \sum(y_C : C\in\mathcal{C}) \\ \text{s.t.} & \sum(y_C : C\in\mathcal{C}, g\in C)\leq 1 \quad \forall\, g\in E \\ & y\geq \mathbf{0}. \end{cases}$$

We will refer to a feasible solution $y\in\mathbb{R}_+^{\mathcal{C}}$ of (D) as a *fractional packing of* $\mathcal{C}$ that has *value* $\sum(y_C : C\in\mathcal{C})$. Since $Q(\mathcal{C})$, the set of feasible solutions of (P), is an integral polyhedron, basic polyhedral theory ensures that a minimum cover of $\mathcal{C}$ yields an optimal solution to (P) (see Theorem 4.1 of [7]). Thus, by Strong LP Duality, an ideal clutter $\mathcal{C}$ has a fractional packing of value $\tau(\mathcal{C})$ (see Theorem 3.7 of [7]); we will need this below.

We first prove that if tangled clutters $\mathcal{A},\mathcal{B}$ have the packing property, then so does $\mathrm{GCP}(\mathcal{A},\mathcal{B})$:

Proof of Theorem 6.2 (i). Take tangled clutters $\mathcal{A},\mathcal{B}$ over ground set E with the packing property. We need to show that $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ has the packing property. By Proposition 1.6 (4), it suffices to show that $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ packs. Before starting the proof, notice that $\mathcal{A},\mathcal{B}$ are ideal by Theorem 1.1, and therefore by Theorem 6.1, $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ is ideal; we will need this fact. Since $b(\mathrm{GCP}(\mathcal{A},\mathcal{B}))=\mathrm{LCL}(b(\mathcal{A}),b(\mathcal{B}))$ by Proposition 1.6 (1), it follows that $\tau(\mathrm{GCP}(\mathcal{A},\mathcal{B}))\geq\tau(\mathcal{A})$ and $\tau(\mathrm{GCP}(\mathcal{A},\mathcal{B}))\geq\tau(\mathcal{B})$.

Claim 1. *If* $\tau(\mathrm{GCP}(\mathcal{A},\mathcal{B}))=\tau(\mathcal{A})$ *or* $\tau(\mathrm{GCP}(\mathcal{A},\mathcal{B}))=\tau(\mathcal{B})$*, then* $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ *packs.*

Proof of Claim. Since every member of $\mathcal{A}$ contains a member of $\mathrm{GCP}(\mathcal{A},\mathcal{B})$, a packing of $\tau(\mathcal{A})$ disjoint members of $\mathcal{A}$ gives a packing of the same number of disjoint members of $\mathrm{GCP}(\mathcal{A},\mathcal{B})$. Hence, if $\tau(\mathrm{GCP}(\mathcal{A},\mathcal{B}))=\tau(\mathcal{A})$, $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ packs. Similarly, if $\tau(\mathrm{GCP}(\mathcal{A},\mathcal{B}))=\tau(\mathcal{B})$, $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ packs, as required. $\Diamond$

Claim 2. *We have either* $\tau(\mathrm{GCP}(\mathcal{A},\mathcal{B}))=\tau(\mathcal{A})$ *or* $\tau(\mathrm{GCP}(\mathcal{A},\mathcal{B}))=\tau(\mathcal{B})$.

Proof of Claim. Suppose otherwise. In particular,

$$2\cdot\tau(\mathrm{GCP}(\mathcal{A},\mathcal{B}))>\tau(\mathcal{A})+\tau(\mathcal{B}).$$

Since $\mathrm{GCP}(\mathcal{A},\mathcal{B})$ is ideal, it has a fractional packing $y\in\mathbb{R}_+^{\mathrm{GCP}(\mathcal{A},\mathcal{B})}$ of value $\tau(\mathrm{GCP}(\mathcal{A},\mathcal{B}))$. We will use this fractional packing to contradict the inequality above. To this end, let A' be a minimum cover of $\mathcal{A}$. Since $\tau(\mathrm{GCP}(\mathcal{A},\mathcal{B}))>\tau(\mathcal{A})$, A' is not a cover of $\mathrm{GCP}(\mathcal{A},\mathcal{B})$, implying in turn that A' is not a cover of $\mathcal{B}$. Since $\mathcal{A}$ entangles $\mathcal{B}$ and A' is not a cover of $\mathcal{B}$, we get that for each $C\in\mathrm{GCP}(\mathcal{A},\mathcal{B})$,

$$|A'\cap C|\geq\begin{cases} 2 & \text{if } C \text{ belongs to } \mathcal{A} \text{ and not to } \mathcal{B} \\ 1 & \text{if } C \text{ belongs to both } \mathcal{A} \text{ and } \mathcal{B} \\ 0 & \text{if } C \text{ belongs to } \mathcal{B} \text{ and not to } \mathcal{A}. \end{cases}$$

Given that $a^\star := \chi_{A'}$, we therefore obtain the following chain of (in)eqaulities:

$$\begin{aligned}\tau(\mathcal{A}) = |A'| &= \sum_{e\in E} a^\star_e \cdot 1\\ &\geq \sum_{e\in E} a^\star_e \cdot \big(y_C : C \in \mathrm{GCP}(\mathcal{A},\mathcal{B}), C \ni e\big)\\ &= \sum_{C\in \mathrm{GCP}(\mathcal{A},\mathcal{B})} y_C \cdot a^\star(C)\\ &\geq 2\cdot \sum \big(y_C : C \in \mathrm{GCP}(\mathcal{A},\mathcal{B}) - \mathcal{B}\big) + \sum \big(y_C : C \in \mathcal{A}, C\in\mathcal{B}\big).\end{aligned}$$

Similarly, we get that

$$\tau(\mathcal{B}) \geq 2\cdot \sum \big(y_C : C \in \mathrm{GCP}(\mathcal{A},\mathcal{B}) - \mathcal{A}\big) + \sum \big(y_C : C \in \mathcal{A}, C\in\mathcal{B}\big).$$

Adding these two together yields

$$\tau(\mathcal{A}) + \tau(\mathcal{B}) \geq 2\cdot \mathbf{1}^\top y = 2\cdot \tau(\mathrm{GCP}(\mathcal{A},\mathcal{B})),$$

yielding the desired contradiction. $\Diamond$

Claims 1 and 2 finish the proof of Theorem 6.2 (i). $\square$

To prove the next part of Theorem 6.2, we need the following ingredient:

Proposition 6.4. *Take tangled clutters $\mathcal{A},\mathcal{B}$ over ground set E, where $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ has the packing property. Let $A\in\mathcal{A}$ be a member that does not contain a member of $\mathcal{B}$, and let $B\in\mathcal{B}$ be a member that does not contain a member of $\mathcal{A}$. Then there are distinct members C,D of $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ such that*

$$C\cap D\subseteq A\cap B \quad \textit{and} \quad C\cup D\subseteq A\cup B.$$

Proof. Let $I := A\cap B$, $J := E-(A\cup B)$ and $\mathcal{C} := \mathrm{LCL}(\mathcal{A},\mathcal{B})/I\setminus J$. Since $A\cup B$ contains a member of $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ and this member is not contained in $A\cap B$, it follows that $\mathcal{C}$ has at least one non-empty member, implying in turn that $\tau(\mathcal{C})\geq 1$. We claim that $\tau(\mathcal{C})\geq 2$. Suppose otherwise. Then there is an element $x\in (A\cup B)-I$ that is contained in every member of $\mathcal{C}$. Take a minimal cover $L\in b(\mathrm{LCL}(\mathcal{A},\mathcal{B})) = \mathrm{GCP}(b(\mathcal{A}),b(\mathcal{B}))$ such that $\{x\}\subseteq L\subseteq \{x\}\cup J$. By the symmetry between $\mathcal{A}$ and $\mathcal{B}$, we may assume that $x\in A-B$ and therefore $L\in b(\mathcal{A})$. As $\mathcal{A}$ entangles $\mathcal{B}$ and A does not contain a member of $\mathcal{B}$, the equation $L\cap A=\{x\}$ implies that L is a cover of $\mathcal{B}$, a contradiction as $L\cap B=\emptyset$. Thus, $\tau(\mathcal{C})\geq 2$. Since $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ has the packing property, $\mathcal{C}$ packs, so it has disjoint members C',D'. Pick members C,D of $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ such that $C'\subseteq C\subseteq C'\cup I$ and $D'\subseteq D\subseteq D'\cup I$. Then C and D are the desired members. $\square$

We are now ready to prove that for tangled clutters $\mathcal{A}$ and $\mathcal{B}$, if $\mathrm{GCP}(\mathcal{A},\mathcal{B}),\mathrm{LCL}(\mathcal{A},\mathcal{B})$ have the packing property, then so do $\mathcal{A},\mathcal{B}$:

Proof of Theorem 6.2 (ii). Take tangled clutters $\mathcal{A},\mathcal{B}$ such that $\mathrm{GCP}(\mathcal{A},\mathcal{B}),\mathrm{LCL}(\mathcal{A},\mathcal{B})$ have the packing property. We need to show that $\mathcal{A},\mathcal{B}$ have the packing property. By Proposition 1.6 (4), it suffices to show that $\mathcal{A}$ and $\mathcal{B}$ pack. Suppose for a contradiction that one of $\mathcal{A},\mathcal{B}$ does not pack. Assuming without loss of generality that $\tau(\mathcal{A})\geq\tau(\mathcal{B})$, we have

$$\tau' := \tau(\mathrm{GCP}(\mathcal{A},\mathcal{B})) \geq \tau(\mathcal{A}) \geq \tau(\mathcal{B}) = \tau(\mathrm{LCL}(\mathcal{A},\mathcal{B})) =: \tau.$$

Since every member of $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ contains a member of $\mathcal{B}$, a packing of τ disjoint members in $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ yields a packing in $\mathcal{B}$ of the same number of members and so $\mathcal{B}$ packs. Thus, $\mathcal{A}$ does not pack and in particular, $\tau(\mathcal{A})>\tau$.

Let $D_1,\ldots,D_{\tau'}$ be a packing of $\mathrm{GCP}(\mathcal{A},\mathcal{B})$. After a possible relabeling, we may assume that

$$\begin{gathered} D_1,\ldots,D_k \text{ do not belong to } \mathcal{B}\\ D_{k+1},\ldots,D_\ell \text{ belong to both } \mathcal{A},\mathcal{B}\\ D_{\ell+1},\ldots,D_{\tau'} \text{ do not belong to } \mathcal{A}, \end{gathered}$$

for some $0\leq k\leq \ell\leq\tau'$. Since $\mathcal{A}$ does not pack and $\tau'>\tau$, it follows that $0<k\leq\ell<\tau'$.

Claim. $k>\tau'-\ell$.

Proof of Claim. Let B' be a minimum cover of $\mathcal{B}$. Notice that B' is also a cover of $\mathrm{LCL}(\mathcal{A},\mathcal{B})$. Suppose for a contradiction that $k\leq\tau'-\ell$. Then it follows from Proposition 6.4 that for each $i\in[k]$, $D_i\cup D_{\ell+i}$ contains two disjoint members of $\mathrm{LCL}(\mathcal{A},\mathcal{B})$, so $D_i\cup D_{\ell+i}$ contains at least two elements of B'. Hence, as $D_{\ell+k},\ldots,D_{\tau'}$ belong to $\mathcal{B}$ and therefore intersect B', we have

$$\tau = |B'| \geq 2k + (\ell-k) + (\tau'-\ell-k) = \tau' > \tau,$$

a contradiction. $\lozenge$

By Proposition 6.4, for each $i\in[\tau'-\ell]$, there are two disjoint members $D_i',D_{\ell+i}'$ of $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ contained in $D_i\cup D_{\ell+i}$. But then since each member of $\mathrm{LCL}(\mathcal{A},\mathcal{B})$ contains a member of $\mathcal{A}$,

$$\begin{aligned} D_i',D_{\ell+i}' &\quad i\in[\tau'-\ell]\\ D_j &\quad \tau'-\ell+1\leq j\leq k\\ D_{k+1},\ldots,D_\ell & \end{aligned}$$

gives rise to a packing in $\mathcal{A}$ of τ' disjoint members, so $\mathcal{A}$ packs, a contradiction. Hence, both $\mathcal{A}$ and $\mathcal{B}$ pack, as required. $\square$

7 Primal clutters

Motivated by the results in the previous two sections, a clutter $\mathcal{C}$ is *not primal* if $\mathcal{C} = \mathrm{GCP}(\mathcal{A}, \mathcal{B})$ or $\mathcal{C} = \mathrm{LCL}(\mathcal{A}, \mathcal{B})$ for some tangled clutters $\mathcal{A}, \mathcal{B}$ neither of which is a lift of the other. A clutter is *primal* if it is *not* not primal. Observe that, for a clutter $\mathcal{C}$, the following statements are equivalent:

- $\mathcal{C}$ is primal,
- if $\mathcal{C} = \mathrm{GCP}(\mathcal{A}, \mathcal{B})$ or $\mathcal{C} = \mathrm{LCL}(\mathcal{A}, \mathcal{B})$ for tangled clutters $\mathcal{A}, \mathcal{B}$, then one of $\mathcal{A}, \mathcal{B}$ is a lift of the other,
- if $\mathcal{C} = \mathrm{GCP}(\mathcal{A}, \mathcal{B})$ or $\mathcal{C} = \mathrm{LCL}(\mathcal{A}, \mathcal{B})$ for tangled clutters $\mathcal{A}, \mathcal{B}$, then one of $\mathcal{A}, \mathcal{B}$ is $\mathcal{C}$.

For instance, the clutters $\{\}, \{\emptyset\}, \big\{\{1\}\big\}$ are primal. Observe that a clutter is primal if, and only if, its blocker is primal.

Proposition 7.1. *For each $n \geq 3$, Δ_n is primal.*

Proof. Take an integer $n \geq 3$ and write $\Delta_n = \big\{\{1,2\}, \{1,3\}, \ldots, \{1,n\}, \{2,3,\ldots,n\}\big\}$. Suppose for a contradiction that $\Delta_n = \mathrm{GCP}(\mathcal{A}, \mathcal{B})$ or $\Delta_n = \mathrm{LCL}(\mathcal{A}, \mathcal{B})$ for some tangled clutters $\mathcal{A}, \mathcal{B}$ over ground set E neither of which is a lift of the other. Since $\Delta_n = b(\Delta_n)$, we may assume by Proposition 1.6 (1) that $\Delta_n = \mathrm{LCL}(\mathcal{A}, \mathcal{B})$. Take new element labels e, f and consider the split join $\mathcal{L} := \mathcal{A}^e \sqcup \mathcal{B}^f$. By Proposition 3.2 (1), e and f are opposite elements, and since neither of $\mathcal{A}, \mathcal{B}$ is a lift of the other, each of e, f is used in at least one member of $\mathcal{L}$. It therefore follows from Proposition 5.9 that there exist disjoint $I, J \subseteq E$ such that $\mathcal{L}' := \mathcal{L} \setminus I/J$ is $\{e,f\}$-simple. That is, there is a partition of $E - (I \cup J)$ into non-empty parts X, Y such that

- $\{e\} \cup X$ and $\{f\} \cup Y$ are the only members of $\mathcal{L}'$ containing either e or f, and
- $\{e\} \cup Y$ and $\{f\} \cup X$ are the only minimal covers of $\mathcal{L}'$ containing either e or f.

Since $\mathcal{A}, \mathcal{B}$ are tangled, it follows from Theorem 4.2 that $\mathcal{L}$, and therefore $\mathcal{L}'$, does not have an (e,f)- or an (f,e)-special minor, implying by Theorem 4.1 that $|X| > 1$ and $|Y| > 1$. Observe that by Proposition 3.2 (3),

$$\mathcal{L}' \setminus e \setminus f = (\mathcal{A}^e \sqcup \mathcal{B}^f) \setminus I/J \setminus e \setminus f = \mathrm{LCL}(\mathcal{A}, \mathcal{B}) \setminus I/J = \Delta_n \setminus I/J.$$

Since $|X| > 1$ and $|Y| > 1$, $\Delta_n \setminus I/J$ has at least two members, X and Y are disjoint minimal covers of it, and so every member of it has cardinality at least two. Thus, $I \cup J \neq \emptyset$, $J = \emptyset$ and $1 \notin I$, and after a possible relabeling, $X = \{1\}$ and $Y = \{2, \ldots, m\}$ for some integer $m \in \{2, \ldots, n-1\}$, a contradiction as $|X| > 1$. $\square$

In fact, other than the trivial examples, the deltas are the only other building blocks of primal clutters!

Theorem 7.2. *Let $\mathcal{C}$ be a clutter different from $\{\}$ and $\{\emptyset\}$. Then the following statements hold:*

(1) If $\mathcal{C}$ has a member $\{e\}$ of cardinality one, then $\mathcal{C}$ is primal if and only if $\mathcal{C} \setminus e$ is primal.

(2) If $\mathcal{C}$ has a minimal cover $\{e\}$ of cardinality one, then $\mathcal{C}$ is primal if and only if $\mathcal{C}/e$ is primal.

(3) If $\mathcal{C}$ has no member or minimal cover of cardinality one, then $\mathcal{C}$ is primal if and only if $\mathcal{C}$ is a delta.

Proof. **(1)** Assume that $\mathcal{C}$ has a member $\{e\}$ of cardinality one. If $\mathcal{C} = \{\{e\}\}$, then the result is obvious. We may therefore assume that $\mathcal{C} \neq \{\{e\}\}$. Suppose first that $\mathcal{C} \setminus e$ is not primal, that is,

$$\mathcal{C} \setminus e = \mathrm{GCP}(\mathcal{A}', \mathcal{B}') \quad \text{or} \quad \mathcal{C} \setminus e = \mathrm{LCL}(\mathcal{A}', \mathcal{B}')$$

for some tangled clutters $\mathcal{A}', \mathcal{B}'$ neither of which is a lift of the other. Let $\mathcal{A} := \{\{e\}\} \cup \mathcal{A}'$ and $\mathcal{B} := \{\{e\}\} \cup \mathcal{B}'$. It can be readily checked that $\mathcal{A}, \mathcal{B}$ are also tangled clutters neither of which is a lift of the other, where either $\mathcal{C} = \mathrm{GCP}(\mathcal{A}, \mathcal{B})$ or $\mathcal{C} = \mathrm{LCL}(\mathcal{A}, \mathcal{B})$, so $\mathcal{C}$ is not primal. Suppose conversely that $\mathcal{C}$ is not primal, that is,

$$\mathcal{C} = \mathrm{GCP}(\mathcal{A}, \mathcal{B}) \quad \text{or} \quad \mathcal{C} = \mathrm{LCL}(\mathcal{A}, \mathcal{B}),$$

for some tangled clutters $\mathcal{A}, \mathcal{B}$ neither of which is a lift of the other. By Proposition 1.6 (4),

$$\mathcal{C} \setminus e = \mathrm{GCP}(\mathcal{A} \setminus e, \mathcal{B} \setminus e) \quad \text{or} \quad \mathcal{C} \setminus e = \mathrm{LCL}(\mathcal{A} \setminus e, \mathcal{B} \setminus e),$$

Moreover, by Remark 1.5, $\mathcal{A} \setminus e$ and $\mathcal{B} \setminus e$ are tangled clutters; it remains to show that these clutters are not a lift of one another. If $\mathcal{C} = \mathrm{LCL}(\mathcal{A}, \mathcal{B})$, then $\{e\}$ is the union of a member of $\mathcal{A}$ and a member of $\mathcal{B}$, implying that $\{e\} \in \mathcal{A}$ and $\{e\} \in \mathcal{B}$, and this in turn implies that $\mathcal{A} \setminus e, \mathcal{B} \setminus e$ are not a lift of one another. Otherwise, $\mathcal{C} = \mathrm{GCP}(\mathcal{A}, \mathcal{B})$. We may assume that $\{e\} \in \mathcal{A}$. Since every minimal cover of $\mathcal{A}$ intersects $\{e\}$ exactly once, and $\mathcal{A}$ entangles $\mathcal{B}$, it follows that either $\{e\} \in \mathcal{B}$ or every minimal cover of $\mathcal{A}$ is also a cover of $\mathcal{B}$. Thus by Proposition 1.2, either $\{e\} \in \mathcal{B}$, or $\mathcal{A}$ is a projection of $\mathcal{B}$, so our hypothesis implies that $\{e\} \in \mathcal{B}$. Thus, $\{e\} \in \mathcal{A}$ and $\{e\} \in \mathcal{B}$, which means that $\mathcal{A} \setminus e, \mathcal{B} \setminus e$ are not a lift of one another, as required. **(2)** follows after applying (1) to the blocker.

(3) By Proposition 7.1, the deltas are primal clutters. Conversely, let $\mathcal{C}$ be a primal clutter over ground set E that has no member or minimal cover of cardinality one. Notice that $b(\mathcal{C})$ is also a primal clutter that has no member or minimal cover of cardinality one. We will show that $\mathcal{C}$ is indeed a delta.

Claim 1. *$b(\mathcal{C})$ does not have disjoint members, and $\mathcal{C}$ does not have disjoint members.*

Proof of Claim. Suppose for a contradiction that $b(\mathcal{C})$ has disjoint members, that is, $\mathcal{C}$ has disjoint minimal covers. In particular, there are covers A, B of $\mathcal{C}$ that partition E. Let $\mathcal{A}$ be the clutter of the minimal sets of $\{A\} \cup \mathcal{C}$, and let $\mathcal{B}$ be the clutter of the minimal sets of $\{B\} \cup \mathcal{C}$. Since $\mathcal{C}$ has no cover of cardinality one, it follows from Proposition 1.4 that $\mathcal{A}, \mathcal{B}$ are tangled clutters neither of which is a lift of the other. However, $\mathcal{C} = \mathrm{LCL}(\mathcal{A}, \mathcal{B})$, a contradiction as $\mathcal{C}$ is primal. Thus, $b(\mathcal{C})$ does not have disjoint members. Similarly, since $b(\mathcal{C})$ is primal and $b(b(\mathcal{C})) = \mathcal{C}$ has no member of cardinality one, $\mathcal{C}$ does not have disjoint members. $\Diamond$

Claim 2. $\mathcal{C} = b(\mathcal{C})$.

Proof of Claim. Since $\mathcal{C}$ does not have disjoint members, every member of $\mathcal{C}$ is also a cover, so every member of $\mathcal{C}$ contains a member of $b(\mathcal{C})$. Similarly, since $b(\mathcal{C})$ does not have disjoint members, every member of $b(\mathcal{C})$ contains a member of $\mathcal{C}$. These two facts together imply that $\mathcal{C} = b(\mathcal{C})$. $\Diamond$

Claim 3. *There do not exist members C_1, C_2 of $\mathcal{C}$ such that $|C_1 - C_2| \geq 2$ and $|C_2 - C_1| \geq 2$.*

Proof of Claim. Suppose otherwise. For each $i \in [2]$, let $B_i := E - C_i$ and $\mathcal{B}_i$ the clutter of the minimal sets of $\{B_i\} \cup \mathcal{C}$. We will prove that $\mathcal{B}_1, \mathcal{B}_2$ are tangled clutters, neither of which is a lift of the other, where $\mathcal{C} = \mathrm{LCL}(\mathcal{B}_1, \mathcal{B}_2)$, thereby contradicting the fact that $\mathcal{C}$ is primal.

First off, $B_1 \cup B_2$ contains a member of $\mathcal{C}$. To see this, take an element $g \in C_1 - C_2$. Since $C_1 \in b(\mathcal{C})$ by Claim 2, there exists a member $C \in \mathcal{C}$ such that $C \cap C_1 = \{g\}$. Notice that $C \subseteq B_1 \cup B_2$. As $B_1 \cup B_2$ contains a member of $\mathcal{C}$, it follows that $\mathcal{C} = \mathrm{LCL}(\mathcal{B}_1, \mathcal{B}_2)$. Secondly, neither $\mathcal{B}_1$ nor $\mathcal{B}_2$ is a lift of the other. To see this, note that $C_1 \in b(\mathcal{C})$ by Claim 2, so $B_1 = E - C_1$ does not contain a member of $\mathcal{C}$. Since B_1 does not contain B_2 either, it follows from Proposition 1.2 (iii) that $\mathcal{B}_1$ is not a lift of $\mathcal{B}_2$. Similarly, $\mathcal{B}_2$ is not a lift of $\mathcal{B}_1$. It remains to show that $\mathcal{B}_1, \mathcal{B}_2$ are tangled. To see why $\mathcal{B}_1$ entangles $\mathcal{B}_2$, take a minimal cover K of $\mathcal{B}_1$ such that $|K \cap B_1| = 1$. It suffices to show that K is also a cover of $\mathcal{B}_2$. Clearly, K intersects every member of $\mathcal{C}$. If $K \cap B_2 = \emptyset$, then $K \subseteq C_2$, so as $C_2 \in b(\mathcal{C})$ by Claim 2, $K = C_2$, implying in turn that $|C_2 - C_1| = |C_2 \cap B_1| = |K \cap B_1| = 1$, which is not the case. Thus, $K \cap B_2 \neq \emptyset$ and subsequently, K is a cover of $\mathcal{B}_2$. Hence, $\mathcal{B}_1$ entangles $\mathcal{B}_2$, and similarly as $|C_1 - C_2| \geq 2$, $\mathcal{B}_2$ entangles $\mathcal{B}_1$, as required. $\Diamond$

Claim 4. *$\mathcal{C}$ is a delta.*

Proof of Claim. Suppose first that $\mathcal{C}$ has a member C of cardinality at least 3. Since C is also a minimal cover by Claim 2, for each $e \in C$, there is another member C_e such that $C_e \cap C = \{e\}$. Since $|C| \geq 3$, we get from Claim 3 that $|C_e| = 2$. Since $\mathcal{C}$ does not have disjoint members by Claim 1, there is an element $g \in E - C$ such that for each $e \in C$, $C_e = \{e, g\}$. Since $\mathcal{C} = b(\mathcal{C})$, it can be readily checked that $\mathcal{C} = \{C\} \cup \{C_e : e \in C\}$, implying in turn that $\mathcal{C}$ is a delta. Suppose in the remaining case that every member of $\mathcal{C}$ has cardinality at most two. Since $\mathcal{C} = b(\mathcal{C})$, we get that every member of $\mathcal{C}$ has cardinality two, and it can be readily checked that $\mathcal{C}$ is in fact a Δ_3, thereby proving the claim. $\Diamond$

Claim 4 finishes the proof of Theorem 7.2. $\square$

8 Identically self-blocking clutters

A clutter $\mathcal{C}$ is *identically self-blocking* if $\mathcal{C} = b(\mathcal{C})$.[9] An identically self-blocking clutter is *trivial* if it is isomorphic to $\{\{1\}\}$, otherwise it is *non-trivial*. The careful reader will notice that these clutters showed up in §3 and §7. The deltas are identically self-blocking clutters. Denote by $\mathbb{L}_7$ the clutter over ground set [7] whose members are the lines of the Fano plane:

$$\mathbb{L}_7 := \big\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,7\},\{2,5,6\},\{3,4,6\},\{3,5,7\}\big\};$$

see Figure 1. $\mathbb{L}_7$ is an identically self-blocking clutter that is binary and non-ideal, as $\begin{pmatrix}\frac{1}{3} & \cdots & \frac{1}{3}\end{pmatrix} \in \mathbb{R}^7$ is an extreme point of $Q(\mathbb{L}_7)$. We have the following easy characterization of identically self-blocking clutters:

[9] In [13] these clutters are called self-blocking clutters. However, it may be more accommodating to reserve that term for clutters that are isomorphic to their blocker, such as $\mathbb{P}_4$.

Remark 8.1 ([5], Theorem 1 on page 46)**.** *Let $\mathcal{C}$ be a clutter different from $\{\}$ and $\{\emptyset\}$. The following statements are equivalent:*

- *$\mathcal{C}$ is identically self-blocking,*
- *$\mathcal{C}$ does not have disjoint members and does not have disjoint covers.*

(Notice that this was shown while proving Theorem 7.2.) Using a so-called *median operator*, Monjardet [18] provided a constructive characterization of all identically self-blocking clutters. Remark 3.4 and Corollary 3.6 give another way to construct all identically self-blocking clutters. By Corollary 3.6, for an arbitrary clutter $\mathcal{A}$, the join $\mathcal{A} \vee b(\mathcal{A})$ is identically self-blocking. Moreover, by Remark 3.4, for an identically self-blocking clutter $\mathcal{C}$ and any element e, we have $(\mathcal{C} \setminus e) \vee b(\mathcal{C} \setminus e) = (\mathcal{C} \setminus e) \vee (\mathcal{C}/e) = \mathcal{C}$.

The *odd hole of dimension* 5 is the clutter over ground set $[5]$ whose members are

$$\mathcal{C}_5^2 := \big\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}\big\}.$$

Notice that $\mathcal{C}_5^2$ is *not* identically self-blocking, and it is non-ideal as $\left(\frac{1}{2}\ \cdots\ \frac{1}{2}\right) \in \mathbb{R}^5$ is an extreme point of $Q(\mathcal{C}_5^2)$. The purpose of this section is to state three conjectures on identically self-blocking clutters:

Conjecture 8.2. *A non-trivial identically self-blocking clutter has one of $\{\mathcal{C}_5^2, \mathbb{L}_7\} \cup \{\Delta_n : n \geq 3\}$ as a minor.*

(The identically self-blocking clutter $\mathcal{C}_5^2 \vee b(\mathcal{C}_5^2)$ has none of $\{\mathbb{L}_7\} \cup \{\Delta_n : n \geq 3\}$ as a minor.) It would be interesting to resolve this conjecture even in the binary case:

Conjecture 8.3. *A non-trivial binary identically self-blocking clutter has an $\mathbb{L}_7$ minor.*

We also make the following weaker conjecture in the general case:

Conjecture 8.4. *A non-trivial identically self-blocking clutter is non-ideal.*

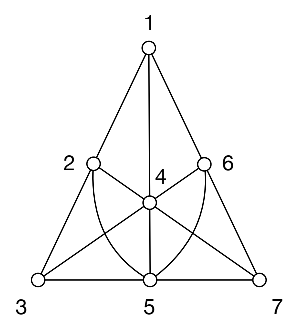

Figure 1: The Fano plane

In the remainder of this section, we prove Conjecture 8.2 in the case when there is a member of cardinality two. We will need the following tool:

Theorem 8.5. *Take a clutter $\mathcal{C}$ and distinct elements e, f such that $\{e, f\}$ is both a member and a minimal cover. Then the following statements are equivalent:*

(i) *there are members C_e, C_f where $C_e \cap \{e,f\} = \{e\}$, $C_f \cap \{e,f\} = \{f\}$ and $C_e \cap C_f \neq \emptyset$,*

(ii) *there exist a member C_e and a minimal cover B_e where $C_e \cap \{e,f\} = \{e\}$, $B_e \cap \{e,f\} = \{e\}$ and $C_e \cap B_e \neq \{e\}$,*

(iii) *there is a delta minor using both e and f.*

Proof. **(i) ⇒ (ii)**: Suppose that (i) holds. Pick an element $g \in C_e \cap C_f$, and choose a minimal cover B_e such that $B_e \cap C_f = \{g\}$. In particular, $f \notin B_e$ and since $B_e \cap \{e, f\} \neq \emptyset$, it follows that $e \in B_e$. Since $C_e \cap B_e \supseteq \{e, g\}$, (ii) holds.

(ii) ⇒ (iii): Suppose next that (ii) holds. Notice that there is no delta minor using exactly one of e, f; it therefore suffices to show the existence of a delta minor using at least one of e, f. We proceed recursively as follows: we either find a delta minor using one of e, f, or we find a proper minor where $\{e, f\}$ is both a member and a minimal cover and (ii) is satisfied. (This process will eventually terminate with a delta minor.) We may assume that $C_e = B_e$, because for $I := C_e - B_e$ and $J := B_e - C_e$, the minor $\mathcal{C}/I \setminus J$ still has $\{e, f\}$ as a member and a minimal cover and satisfies (ii). Let C_f be a member such that $C_f \cap \{e, f\} = \{f\}$. Note that $C_f \cap C_e = C_f \cap B_e \neq \emptyset$. If $C_f - \{f\} \subseteq C_e$, then $\mathcal{C}$ has a delta minor using f by applying Theorem 5.4 to the members $\{f, e\}, C_f, C_e$. We may therefore assume that $X := (C_f - \{f\}) - C_e \neq \emptyset$. If there is a member C contained in $\{e\} \cup X$, then $\mathcal{C}$ has a delta minor using e by applying Theorem 5.4 to the members $\{e, f\}, C, C_f$. Thus we may assume that no member is contained in $\{e\} \cup X$. Consequently, $\{e, f\}$ is a member and a minimal cover for the minor $\mathcal{C}' := \mathcal{C}/X$. Let C'_e be a member of $\mathcal{C}'$ contained in C_e; as $C'_e \cap \{e, f\} \neq \emptyset$ we have $e \in C'_e$. But since B_e is a minimal cover of $\mathcal{C}'$ and $C'_e \cap B_e \neq \{e\}$, it follows that $\mathcal{C}'$ satisfies (ii), so we can recurse. This shows that (iii) holds.

(iii) ⇒ (i): Suppose finally that (iii) holds, that is, there are disjoint element subsets I, J such that $(I \cup J) \cap \{e, f\} = \emptyset$ and $\mathcal{C}/I \setminus J$ is a delta. Clearly, $\{e, f\}$ is both a member and a minimal cover of this minor, so there are two other members C'_e, C'_f of this minor such that $C'_e \cap \{e, f\} = \{e\}$, $C'_f \cap \{e, f\} = \{f\}$, and since this minor is a delta, we have that $C'_e \cap C'_f \neq \emptyset$. Now let C_e, C_f be members of $\mathcal{C}$ such that $C'_e \subseteq C_e \subseteq C'_e \cup I$ and $C'_f \subseteq C_f \subseteq C'_f \cup I$. Observe that $C_e \cap C_f \supseteq C'_e \cap C'_f \neq \emptyset$, implying (i). □

As a consequence,

Corollary 8.6. *An identically self-blocking clutter with a member of cardinality two has a delta minor.*

Proof. Let $\mathcal{C}$ be an identically self-blocking clutter with a member of the form $\{e, f\}$. Since $\mathcal{C} = b(\mathcal{C})$, $\{e, f\}$ is also a minimal cover. So there are members C_e, C_f such that $C_e \cap \{e, f\} = \{e\}$ and $C_f \cap \{e, f\} = \{f\}$. By Remark 8.1, it follows that $C_e \cap C_f \neq \emptyset$, so by Theorem 8.5, $\mathcal{C}$ has a delta minor. □

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